

5

Electric potential and gradient



Fig.5.4.4, Source: By "Nitromethane" - Own work, CC BY-SA 3.0. Description: Corona discharge on a corona ring of 500 kV overhead power line. The picture was taken at night with 90 s exposure time and ISO 200. The corona discharge looks weaker to the naked eye, but the long exposure time does the job. See: <https://commons.wikimedia.org/w/index.php?curid=27344360>

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5.1 Basic definition of electric potential

Electric potential is defined as the electric potential energy per charge:

$$V = \frac{U_{\text{e pot}}}{q_0} \quad \text{eq. 5.1.1}$$

The unit is Volt = V = J / C.

The electric potential is the electric potential energy associated with a test charge q_0 at a specific point divided by the test charge q_0 itself.

It depends of course on where q_0 is placed. If we place q_0 at another point, the electric potential energy, and hence the electric potential, will in general be different. The test charge q_0 has to exist in order to have an electric potential energy in the same way as q_0 should exist in order to have an electric force acting on q_0 .

However the electric potential (and the E-field) will also exist if we have no test charge because the electric potential (and the E-field) at a specific point is the electric potential energy per charge (or force per charge for the E-field) that *would* be there if we place a charge at that point. If you have access to the web-site, it is now recommended to bring chapter 4 in the right frame.

In section 4.4 we derived a formula for the electric potential energy of q_0 relative to some other charges $q_1, q_2, q_3, \dots$:

$$U_{\text{e pot } q_0} = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_0 q_1}{r_{01}} + \frac{q_0 q_2}{r_{02}} + \frac{q_0 q_3}{r_{03}} + \dots \right\} = \frac{q_0}{4\pi\epsilon_0} \sum_i \frac{q_i}{r_{0i}} \quad \text{eq. 4.4.6}$$

Here, r_{0i} is the distance from q_0 to q_i , see for example fig.5.1.1 where $q_1 = q_2 = q_3$.

$U_{\text{e pot } q_0}$ means the electric potential energy of q_0 relative to other charges. It depends on q_0 's charge and q_0 's position relative to the other charges.

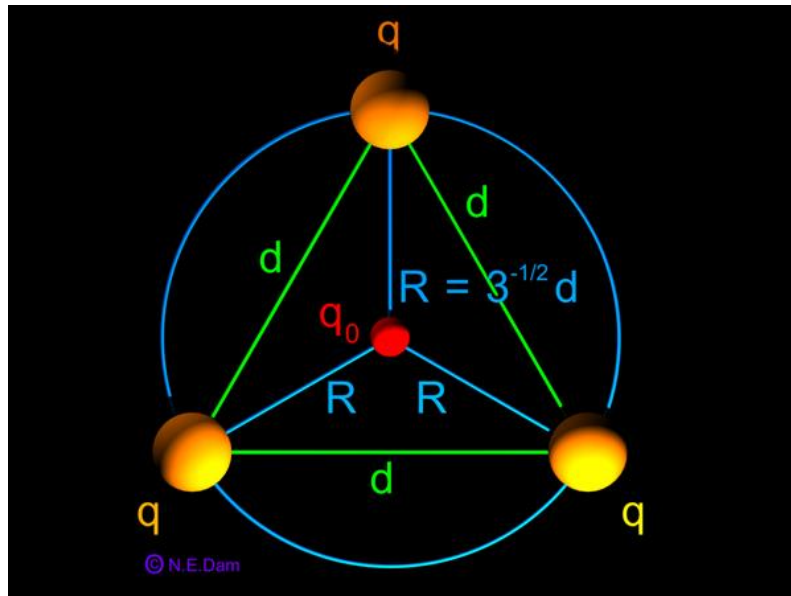


Fig.5.1.1, ref.1 (N.E. Dam ©). A molecule with an extra (red) ion placed in the center.

This formula may be divided by q_0 to get the electric potential at q_0 's position:

$$V_{q_0} = \frac{U_{\text{e pot } q_0}}{q_0} = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_1}{r_{01}} + \frac{q_2}{r_{02}} + \frac{q_3}{r_{03}} + \dots \right\} = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{r_{0i}} \quad \text{eq. 5.1.2}$$

The electric potential at q_0 's position may be calculated either by using eq.4.4.6 and then divide by q_0 or directly by using the middle part or the right side of eq.5.1.2.

If we have a collection of charges, eg. like in fig.5.1.2, we may calculate the electric potential energy for the collection of charges using a formula from chapter 4:

$$U_{\text{e pot}} = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} + \dots \right\} = \frac{1}{4\pi\epsilon_0} \sum_{i < j} \frac{q_i q_j}{r_{ij}} \quad \text{eq. 4.4.4}$$

However, in general we cannot define the electric potential for a collection of charges. The electric potential is namely defined as the electric potential energy of a *single* test charge q_0 (not a collection of charges!) divided by the same test charge q_0 . Here, q_0 should be infinitesimal small in order not to disturb the charge distribution.

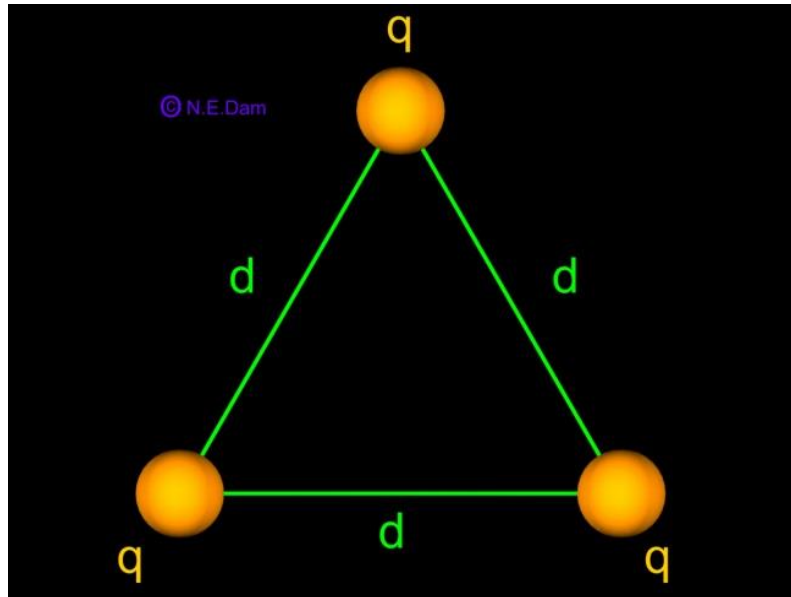


Fig.5.1.2, ref.1 (N.E. Dam ©). A triangular molecule with 3 ions (orange) placed in the corners.

When an electric force $\vec{F}$ moves q_0 from a to b, see for example fig.5.2.1, we may calculate the work done by $\vec{F}$ using eq.4.2.3 and eq.4.2.4. It gives:

$$W_{ab} = \int_a^b \vec{F} \cdot d\vec{l} = q_0 \int_a^b \vec{E} \cdot d\vec{l} = U_{e\text{ pot, a}} - U_{e\text{ pot, b}} \quad \text{eq. 5.1.3}$$

The work per unit charge is then:

$$V_{ab} = \frac{W_{ab}}{q_0} = \int_a^b \vec{E} \cdot d\vec{l} = \frac{U_{e\text{ pot, a}}}{q_0} - \frac{U_{e\text{ pot, b}}}{q_0} = V_a - V_b \quad \text{eq. 5.1.4}$$

V_{ab} is called the voltage drop (or the potential drop) from a to b.

It is normally measured by a voltmeter. If we know the electric field, the voltage drop may be calculated using the integral, which is independent of the integration path. If we have a closed integration path where $a=b$, we get

$$\oint \vec{E} \cdot d\vec{l} = 0 \quad \text{eq. 5.1.5}$$

In other words the integral of $\vec{E}$ along a closed path is zero for a normal electrostatic E-field, which is conservative. Eq.5.1.5 may be proven for a Coulomb field.

Now, it is recommended to consider the problems 5.1 and 5.2.

5.2 The potential drop in a uniform E-field

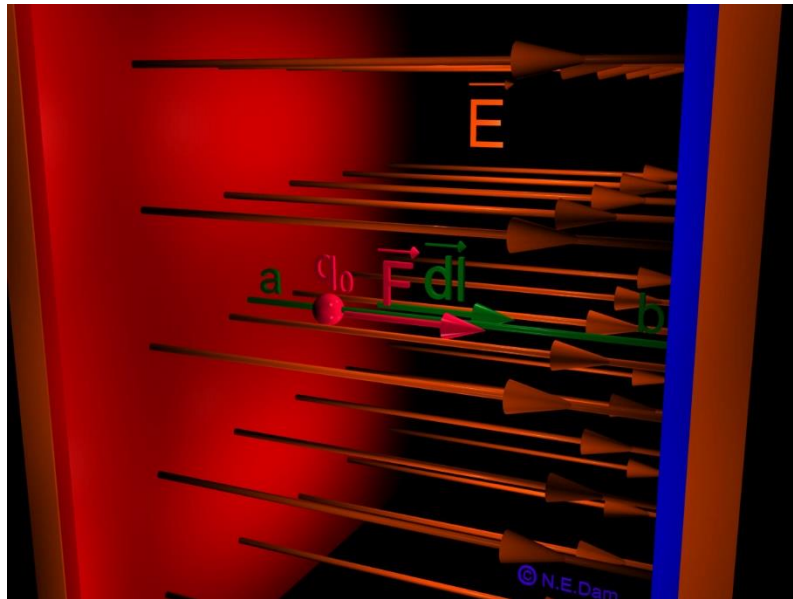


Fig.5.2.1, ref.1 (N.E. Dam ©). A charge q_0 moving inside a double layer from a to b with a uniform E-field (e.g. in a capacitor).

Fig.5.2.1 shows a so-called double layer. It consists of two parallel plates (layers), the one layer is positively charged and the other layer is negatively charged. The E-field is uniform and perpendicular to the two layers.

If a charge q_0 is moving in such an E-field from a to b in fig.5.2.1 we may use eq.5.1.3 to calculate the work W_{ab} done by the electric force acting on q_0 . Then we may divide the result by q_0 to find the potential drop V_{ab} .

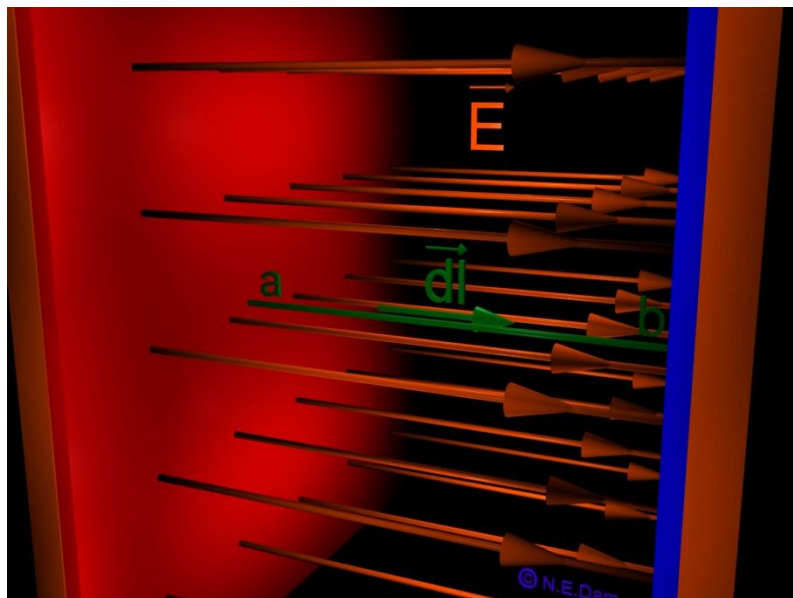


Fig.5.2.2, ref.1 (N.E. Dam ©). The potential drop from a to b may be calculated as the integral of $\vec{E} \cdot \vec{dl}$ from a to b.

If we are not interested in knowing the work W_{ab} we may use eq.5.1.4 to find V_{ab} directly, see also fig.5.2.2:

$$V_{ab} = \int_a^b \vec{E} \cdot d\vec{l} = \int_a^b E dl = E \int_a^b dl = E d = V_a - V_b \Rightarrow E = \frac{V_a - V_b}{d} = \frac{V_{ab}}{d} \text{ eq. 5.2.1}$$

That is, a uniform E-field may be calculated as the electric potential drop divided by the distance d . Therefore, the unit for the E field is not only N/C but also V/m.

How a positive charge will move in an E-field

We note, that the electric force is in the same direction as the E-field, see fig.5.2.1. If a positive charge is moving in the direction of the E-field, it will move in the direction of the force, then W_{ab} and $V_a - V_b$ is positive, i.e. $V_a > V_b$. Moving in the direction of the E-field means that the electric potential V decreases. It also means, that a positive charge would like to move in the direction of the E-field towards a lower electric potential. It "falls down" towards a lower electric potential just like a ball rolls downhill towards a lower potential energy. At the same time, the kinetic energy increases or heat is developed. Moving in the opposite direction of the E-field means that V increases.

How a negative charge will move in an E-field

We note, that the electric force is in the opposite direction of the E-field, see fig.5.2.3. A negative charge would therefore like to move in the opposite direction of the E-field because the electric force is in that direction. According to eq.5.1.3, W_{ab} is still positive, because the force $\vec{F}$ is in the same direction as the direction of the movement $d\vec{l}$. According to eq.5.2.1, $V_{ab} = V_a - V_b$ is negative, because $d\vec{l}$ is in the opposite direction of $\vec{E}$. It means that $V_a < V_b$, the negative charge moves towards a higher potential. However, the negative charge moves towards a lower potential energy.

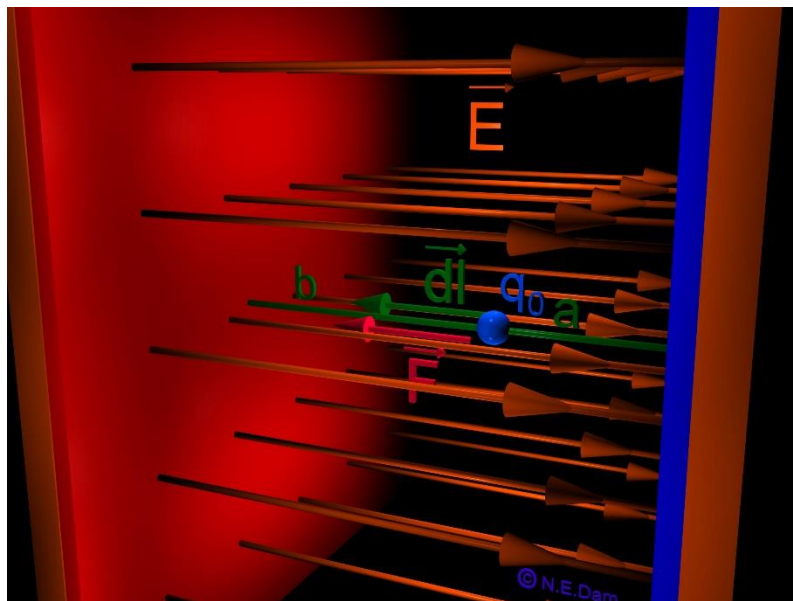


Fig.5.2.3, ref.1 (N.E. Dam ©). A negative charge $q_0 < 0$ moving inside a double layer from a to b.

Definition of an eV

We consider a test charge $q_0 = 1e = 1,602 \cdot 10^{-19} \text{ C}$ moving through an electric potential drop of 1 V. According to eq.5.1.4 the work done by the electric force is

$$W_{ab} = q_0 V_{ab} = 1e \cdot 1V = 1 \text{ eV} = 1,602 \cdot 10^{-19} \text{ C} \cdot 1 \text{ J/C} = 1,602 \cdot 10^{-19} \text{ J} \quad \text{eq.5.2.2}$$

In other words, if a particle of charge e moves through an electric potential drop of 1 V, the drop in electric potential energy (or the increase in kinetic energy) is 1 eV.

This energy unit is useful when we consider energy levels in atoms or molecules or energy levels of electrons in semiconductors etc.

If we, as an example, measure a voltage drop of 2 V across a LED, it means, that if a hole with the charge e moves across the PN boundary, the drop in electric potential energy for the hole will be 2 eV.

The unit eV is also used when charged particles are accelerated in an accelerator, see fig.5.2.4.



*Fig.5.2.4, ref. Argonne National Laboratory
A linear accelerator is used to accelerate electrons to almost the speed of light. Then the electrons may be injected into a so-called synchrotron, where the electrons are accelerated further.*

Now, it is recommended to consider problem 5.3.

5.3 Electric potential outside and inside a charged conducting sphere

Fig.5.3.1 shows a solid conducting sphere of radius R and charge Q . The curve $E(r)$ is shown in fig.5.3.2. It was derived in chapter 3.

We want to find the electric potential $V(r)$.

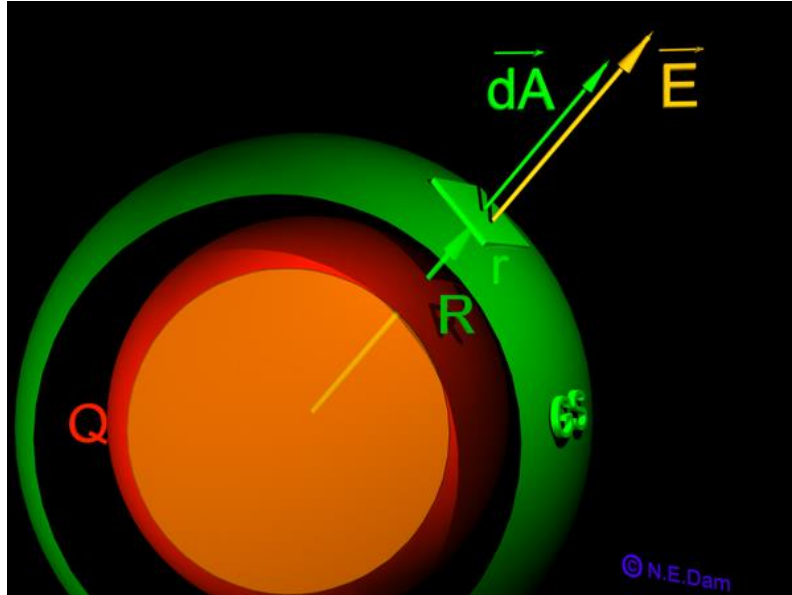


Fig.5.3.1, ref.1 (N.E. Dam ©). Calculating the electric potential outside and inside a conducting sphere.

Outside the sphere

The electric field was calculated in section 3.6.2 using Gauss's law. Now we calculate the electric potential outside the sphere using eq.5.1.4 and the limits r and ∞ (where the electric potential = 0)

$$V_{ab} = \int_r^\infty \vec{E} \cdot d\vec{l} = V_r - V_\infty = V_r \Rightarrow V_r = \int_r^\infty \vec{E} \cdot d\vec{l} = \frac{Q}{4\pi\epsilon_0} \int_r^\infty \frac{1}{r^2} dr \Rightarrow$$

$$V_r = \frac{Q}{4\pi\epsilon_0} \frac{1}{r} \quad \text{eq. 5.3.1}$$

Inside the sphere

We integrate from a point r inside the sphere to a point on the surface:

$$V_{ab} = \int_r^R \vec{E} \cdot d\vec{l} = V_r - V_R = 0 \quad \text{eq. 5.3.2}$$

since $E = 0$ inside the sphere. Hence

$$V_r = V_R = V_Q = \frac{q}{4\pi\epsilon_0} \frac{1}{R} \quad \text{eq. 5.3.3}$$

ie. the electric potential remains constant at the same value as it has on the surface.

Fig.5.3.3 shows $V(r)$ inside and outside the conducting sphere. Please note, that $V(r)$ is constant and not necessarily zero inside the conducting sphere. In fact, it is not just a conducting sphere, where the potential is constant inside the material. It is true for any conductor no matter the shape.

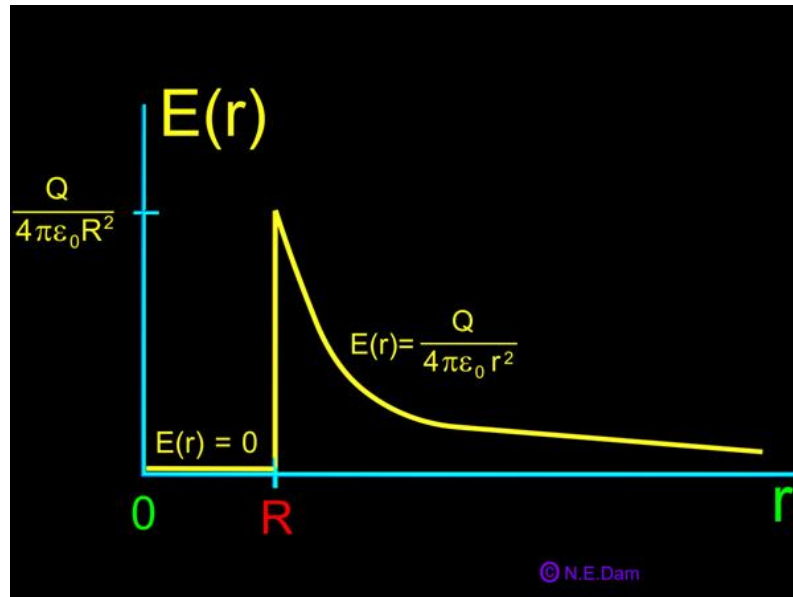


Fig.5.3.2, ref.1 (N.E. Dam ©).
 $E(r)$ for the conducting sphere.

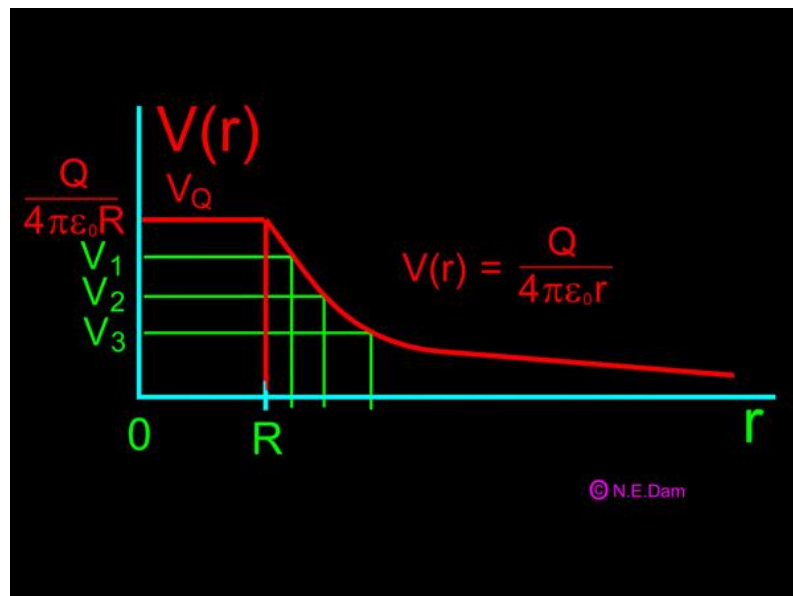


Fig.5.3.3, ref.1 (N.E. Dam ©).
 $V(r)$ for the conducting sphere.

5.4 Corona discharge

Introduction

The corona discharge is a discharge phenomenon close to a sharp corner of a high voltage conducting object. If we consider a conducting object with a constant electric potential, the E-field outside a sharp corner is much stronger than the E-field outside a flat surface. The E-field close to such a sharp corner may be strong enough to ionize some of the air molecules. This causes the air to “glow” in a bluish colour close to the corner as shown in fig.5.4.2-4. In order to explain the phenomenon in more detail we consider the following subjects:

- The E-field close to the surface of a sphere
- The E-field at a variable distance from the surface of a sphere
- The corner as a part of a sphere
- The corona discharge outside a corner.

The E-field close to the surface of a sphere

Just outside the surface of the conducting sphere in fig.5.3.1-3 we have:

$$E_R = \frac{Q}{4\pi\epsilon_0} \frac{1}{R^2} \quad \text{eq. 5.4.1}$$

and

$$V_R = \frac{Q}{4\pi\epsilon_0} \frac{1}{R} \quad \text{eq. 5.4.2}$$

Hence, we find for the E-field just outside the surface of the conducting sphere:

$$E_R = \frac{V_R}{R} \quad \text{eq. 5.4.3}$$

The E-field in eq.5.4.3 is only valid close to the surface (outside the surface).

The E-field at a variable distance from the surface of a sphere

A general expression for the E-field outside a conducting sphere is derived in chapter 3 using Gauss’s law. The result is the E-field as a function of the distance r from the center of the sphere to the field point.

$$E_r = \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \quad \text{eq. 5.4.4}$$

This expression for the E-field is valid at any distance outside the conducting sphere.

The corner as a part of a sphere

The corner in fig.5.4.1 may be considered as $\frac{1}{8}$ of a sphere. The radius of this sphere is denoted as the radius of curvature (Dan: krumningsradius) for the corner.

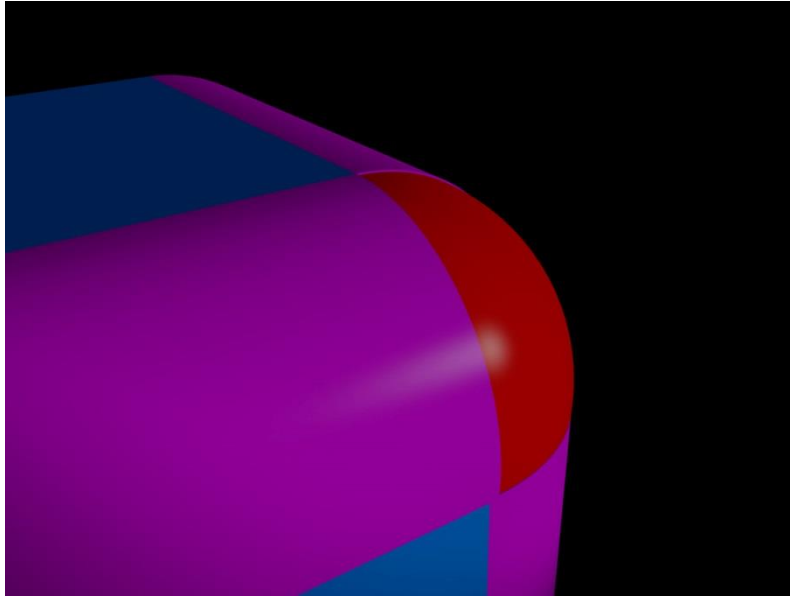


Fig.5.4.1, ref.1 (N.E. Dam ©). Different parts of a corner. The flat surfaces are shown in a blue colour. Each of the edges may be considered as 1/4 of a purple cylinder. The corner itself is considered as 1/8 of a red sphere.

The E-field just outside the corner, E_R , depends on the radius R of the sphere (the radius of curvature), see eq.5.4.3. We now consider a metallic object with such a corner. The electric potential of the object, V_R , is kept constant. If the radius R decreases, the E-field just outside the corner, E_R , increases. If the radius R becomes less than the so-called *critical radius of curvature*, R_c , the E-field E_R exceeds the so-called *critical E-field*, E_c . The critical value E_c depends on the gas in the atmosphere outside the corner as well as the temperature and pressure.

The corona discharge outside a corner

When $E_R > E_c$, the gas molecules become ionized, i.e. electrons are removed from many of the gas molecules. The ionized molecules are referred to as excited molecules because each of them has lost one or more electrons. When an electron returns to an excited molecule, light is emitted. An ionized gas is also denoted as a plasma. The gas is usually ionized in channels, the so-called *leaders* or *plasma filaments* see fig.5.4.2-3.

Other examples are the corona discharges shown in fig.5.4.3 and in fig.5.4.4.

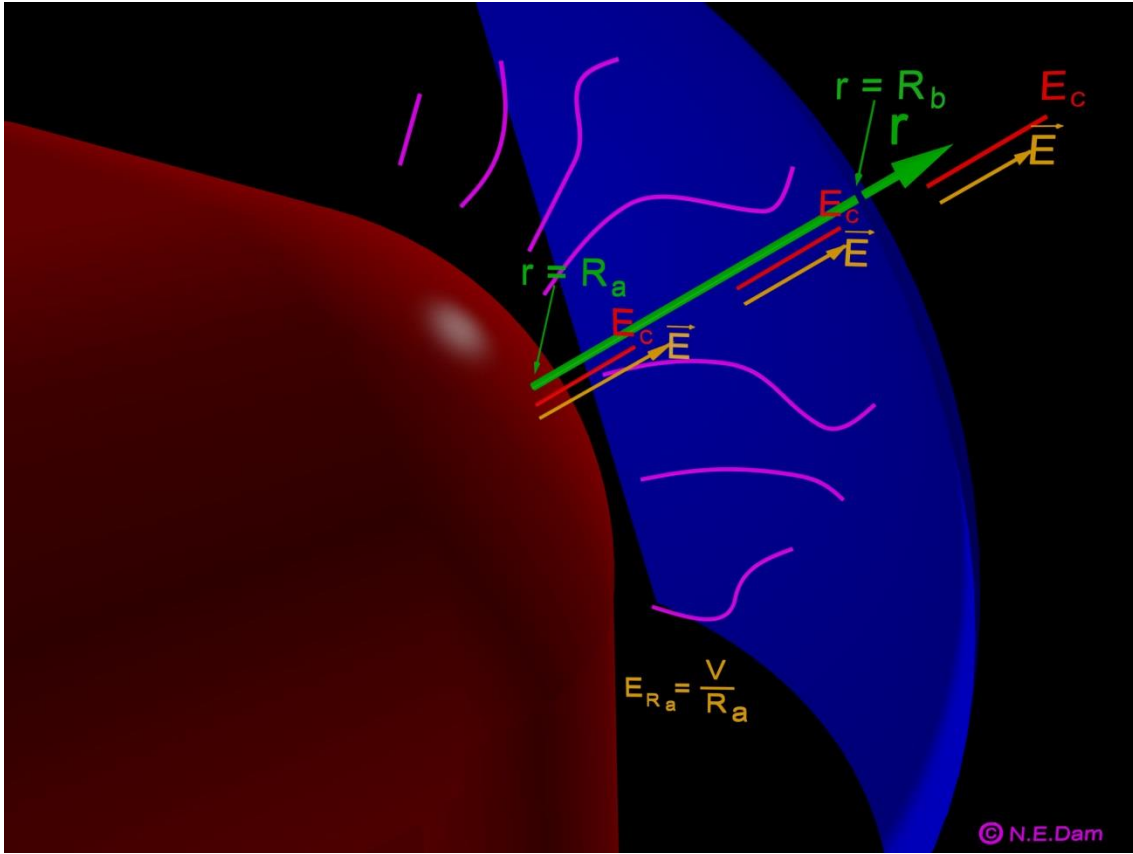


Fig.5.4.2, ref.1 (N.E. Dam ©). Close to a corner (inside the blue surface), the E-field magnitude exceeds the critical value E_c . This causes the atmosphere to be ionized and the so-called plasma filaments (purple) are formed. At a further distance (outside the blue surface) the E-field magnitude is less than the critical value E_c and no plasma filaments are formed here.

Equation 5.4.3 may be used to calculate the critical radius of curvature R_c if we know the voltage of a high voltage equipment and the critical field. It means that if the radius of curvature $R < R_c$, the E-field close to the surface is larger than the critical field for the atmosphere around the equipment. The result will be a localized ionizing of the atmosphere just outside the surface, the so-called *corona discharge* shown in fig.5.4.3-4.

As a repetition, eq.5.4.3 shows that if we have a corner with a small radius of curvature ($R < R_c$), we have a very high E-field magnitude just outside the surface of that corner, see fig.5.4.2. From $r = R_a$ to $r = R_b$, this E-field exceeds the critical E-field, E_c , which is necessary for ionizing the air just outside the corner. The plasma filaments light up and causes the air to “glow” just outside the corner. At a further distance from the corner ($r > R_b$), the E-field is below what is necessary for ionizing the air. *Therefore the corona discharge is a localized phenomenon close to a corner.*

Fig.5.4.2 shows, that between the corner and the blue surface, the E-field exceeds E_c and we get the plasma filaments and the corona discharge.

Now, it is recommended to consider problem 5.4.

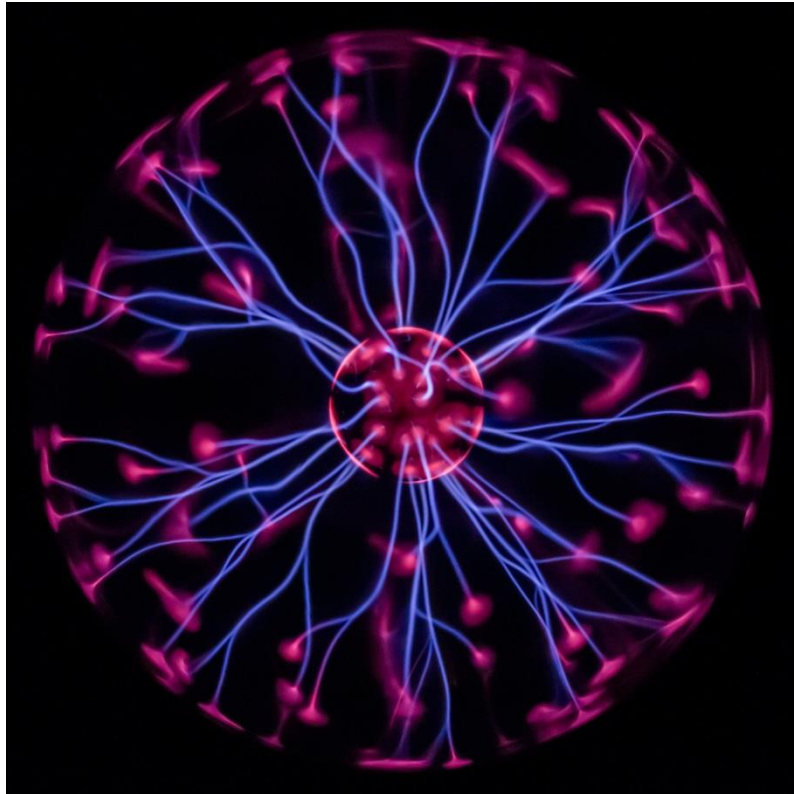


Fig.5.4.3, Plasma filaments in a plasma globe. The filaments extend from the inner sphere towards the outer sphere, which is a glass sphere covered by a thin layer of graphite on the inner side. © Colin. <https://commons.wikimedia.org/w/index.php?curid=30044004>



Fig.5.4.4, Source: By "Nitromethane" - Own work, CC BY-SA 3.0. Description: Corona discharge on a corona ring of 500 kV overhead power line. The picture was taken at night with 90 s exposure time and ISO 200. The corona discharge looks weaker to the naked eye, but the long exposure time does the job. See: <https://commons.wikimedia.org/w/index.php?curid=27344360>

5.5 Arc formation



Fig.5.5.1 Arc formation between two electrodes, where the electric potential difference is 3 kV.

*By Achim Grochowski - Achgro - Own work, CC BY 3.0,
<https://commons.wikimedia.org/w/index.php?curid=202817>*

An electric arc

An electric arc (Dan lysbue) is when the air between two electrodes becomes a conducting plasma, see fig.5.5.1. It occurs when the E-field between the two electrodes is higher than the critical E-field E_c , i.e. when $E > E_c$ all the way from the one electrode to the other electrode. The arc or plasma causes the air to glow between the two electrodes. The plasma is extremely hot and has therefore a less density than the surrounding air. Hence, the arc bows upward because of the buoyant effect (Dan opdrift) as illustrated in fig.5.5.1. Another detail is, that an arc is considered to be ongoing, not a very short event like a spark or a lightning.

The difference between corona discharge and arc formation

The corona discharge is close to a conducting object with a sharp corner and $E > E_c$ only within a relative short distance from the corner. In contrast to this, the arc formation involves two electrodes and $E > E_c$ all the way from the one electrode to the other. Hence, it requires a relatively high potential difference between the two electrodes to start the arc formation. However, once started the arc, the plasma between the two electrodes is highly conductive and a much lower potential difference is required to maintain the plasma.

Brief explanation of plasma

A plasma is an ionized gas with positive and negative ions. In total, the overall charge of a plasma is almost zero. When the plasma is formed, atoms in the gas are ionized and become positive atomic ions. Each atomic ion consists of a nucleus with some electrons. Here, the number of electrons is less than the number of protons in the nucleus. Hence, the net charge of an atomic ion is positive. The remaining electrons are excited and act almost like conduction electrons in a metal giving the plasma a very high conductivity. The positive atomic ions are “swimming” around in a “sea” of conduction electrons. In a metal, the positive atomic ions are positioned in a crystal lattice. In a plasma gas the positive atomic ions are moving around in a completely disordered way.

A plasma may be generated either in a gas where $E > E_c$ within some volume or in gas which is heated to a very high temperature. The plasma in a lightning channel or the other plasmas considered in this section are only partly ionized, i.e. each atomic ion has electrons orbiting around the nucleus and only a part of the electrons are excited away from the atomic ion. Inside the sun there is a completely ionized plasma, where all electrons are excited.



Fig.5.6.1 © Mike Sharp, Posted to The Cloud Appreciation Society on August 31, 2012. A large waterspout illuminated by lightning, spotted during a storm that passed through the Bahama Islands.

5.6 Practical applications of arc formation

The thunderstorm

Lightning occurs in connection with the thunderstorm, see fig.5.6.1. Strictly, the lightning is not considered to be an arc, since it is not ongoing. However, it is mentioned in this section. If the underside of the thundercloud (Latin: Cumulonimbus) is negatively charged, the earth surface becomes positively charged by electrical induction almost as explained in chapter 2, see fig.2.3.4. When the electric field between the surface of the earth and the underside of the cloud becomes sufficiently large, the air is ionized and very many plasma filaments are formed, especially from metal rods at the top of antennas etc. These plasma filaments may be visible if it is dark enough because they only light up a little. Later, some of the plasma filaments are connected and used as a lightning channel when the discharge process (between the earth or the water surface and the underside of the thundercloud) becomes active. The current is typically in the range 50 – 100 kA.

Arcing above electric trains

We see the arc formation at all times during the year but especially in the winter time, when ice has covered the overhead wire, which is above electric trains. The overhead wire is also called a contact wire or a catenary wire (Dan luftledning over et tog). If parts of the wire are covered by ice, the ice acts like a switch turning the current on and off. When the current is turned off, the highly inductive engine induces a very high voltage between the catenary wire and the pantograph (Dan strømaftager på taget af det elektriske tog). An arc is formed between the catenary wire and the pantograph, see fig.5.6.2. It lights up the environment with a strong bluish light. In addition to this, the arc also produces ozone, which can be smelled in a tunnel with a heavy traffic of electric trains. Even without ice on the catenary wire, arcs are formed at many places along the catenary wire.



*Fig. 5.6.2 Arcing above a train at the Gare de Nord.
By Mihai Călin, Videoclip: <https://youtu.be/tYjIRgMHwpI>*

Arcing at electric breakers

Arcs are also formed at breakers in outdoor high voltage equipment, see the videoclip at <https://youtu.be/hIkNY5xjy5k>. When the breaker brakes the current, the circuit induces a voltage, which is high enough to start an arc formation between the contacts in the breaker. Once started, the arc may continue to some extent at a lower E-field, when the contact arms are turned away from each other.

The phenomenon may be harmful, because it breaks down insulators etc. Hence, it is an important issue to construct breakers which minimises the arc formation or minimises the destructive effect on the equipment.

Welding

An electric arc is used when welding metal objects together, e.g. construction of buildings and ships. An electric arc may also be used to cut in metal objects.

Melting metal

A method to melt metal is to lower a very big electrode down into a very large container with metal, see fig.5.6.3.



*Fig.5.6.3 By Deutsche Fotothek, CC BY-SA 3.0 de,
<https://commons.wikimedia.org/w/index.php?curid=6389626>*

Arc lamps

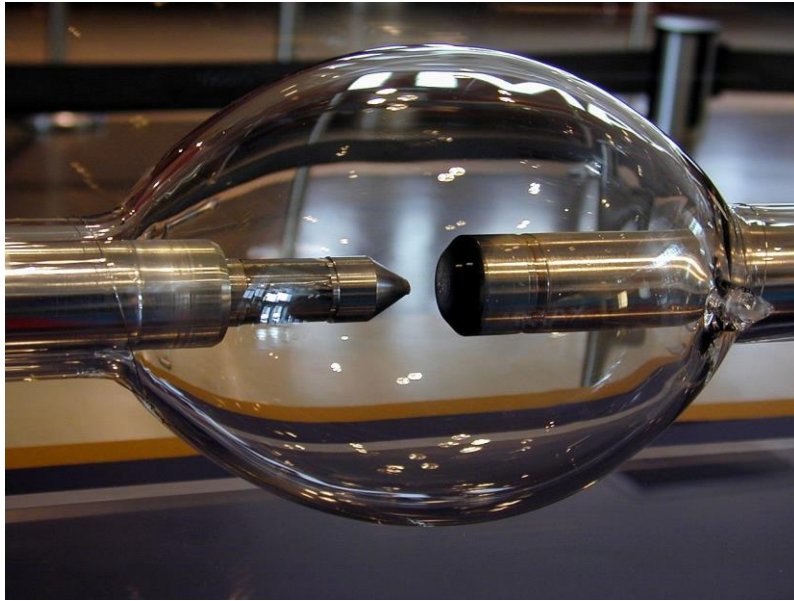


Fig.5.6.4 The 15 kW xenon short-arc lamp used in the IMAX projection system. By Atlant, CC BY 2.5, <https://commons.wikimedia.org/w/index.php?curid=294010>

Very strong light is produced by arc formation in evacuated bulbs with two electrodes, see fig.5.6.4. The electrodes become very hot, more than 1000 K. Electrons and – to some extent also atoms – are emitted from the surface of the negative electrode in almost the same way as water molecules are evaporated from boiling water. The lifetime of such bulbs is limited to some hundreds of hours. Such bulbs are used in lighthouses, cinema projectors etc.

5.7 Light sources using plasma glow

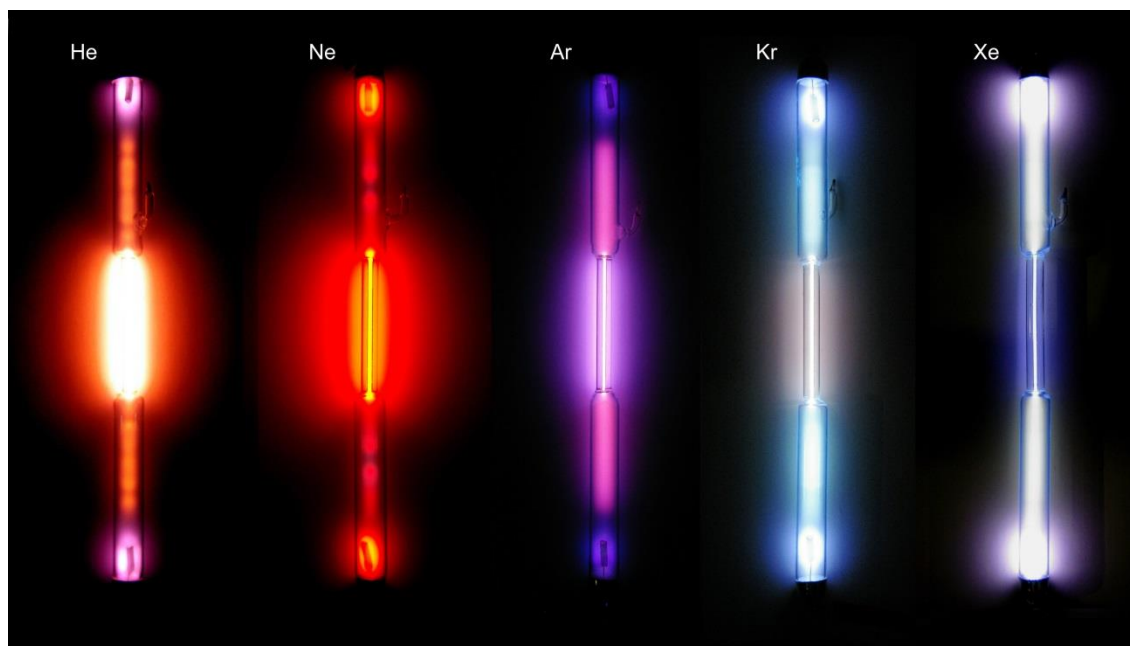


Fig.5.7.1 Spectrum = gas discharge tubes: the noble gases: helium He, neon Ne, argon Ar, krypton Kr, xenon Xe. Used with 1,8kV, 18mA, 35kHz. $\approx 8''$ length.

By Alchemist-hp www.pse-mendelejew.de - Own work, CC BY-SA 2.0 de, <https://commons.wikimedia.org/w/index.php?curid=7601144>

Introduction

Many of such light sources are produced by evacuating a glass tube and then adding a small amount of mercury (Hg) or sodium (Na) or a noble gas such as helium (He), neon (Ne), argon (Ar), krypton (Kr), xenon (Xe).

There are two electrodes in such a glass tube, one at each end, see fig.5.7.1. A voltage of several kV is applied across the electrodes causing an arc formation from the one end of the glass tube to the other end. The resistance of the plasma in such a tube is much larger than the resistance of the plasma in the arc lamp described in the previous section. Hence, the current through the plasma in the tube is much lesser than in the arc lamp. However, the relatively small current in the tube causes the plasma to “glow” because the atoms in the gas in the tube are excited. When the atoms return to the ground state, they emit light with a characteristic wavelength or frequency corresponding to the energy level in the excited electron orbit. Different atoms have different different energy levels for the electrons and this causes different colours, see fig.5.7.1.

The electrodes in the tubes are “cold” (not heated). The plasma is supplied with electrons from the cold cathode. The E-field in the tube is relatively high and electrons are emitted from the cold cathode.

An electron avalanche is a process in which a number of free electrons in a transmission medium are subjected to strong acceleration by an electric field and subsequently collide with other atoms of the medium, thereby ionizing them (impact ionization).

This releases additional electrons which accelerate and collide with further atoms, releasing more electrons causing a chain reaction. In a gas, this causes the affected region to become an electrically conductive plasma. Brief explanations follow in the subsequent sections.

Neon Tubes



*Fig.5.7.2 Four neon tubes put together to show the sign Ne.
By Pslawinski, CC BY-SA 2.5,
<https://commons.wikimedia.org/w/index.php?curid=381579>*

A neon tube consists of an evacuated glass tube with two electrodes, one at each end and then filled with neon at low pressure. A high voltage of several kV is applied across the two electrodes causing an arc formation in the neon gas. It is not an arc with a strong current as described above in the arc lamps, it is an arc with a much smaller current flow and the plasma in the tube has a much higher resistance compared to the relative short and intense arc in the arc lamps.

The neon tubes are still very much used for commercial signs and different colours can be obtained by using different noble gasses such as helium (He), neon (Ne), argon (Ar), krypton (Kr), xenon (Xe).

Neon lamps

The neon lamps can be less than one centimeter in length and glow much more dimly than the tube lights. They are still in use as small indicator lights, see fig.5.7.3. Earlier, neon glow lamps were widely used for numerical displays in electronic equipments, see fig.5.7.4. These lamps are now antiques. Very small neon glow lamps have been used as pixels in TV screens.



Fig.5.7.3 Close-up view of a 5mm diameter NE-2 type neon lamp. Both electrodes appear to glow simultaneously due to being powered by alternating current, with the exposure time being longer than a full current cycle. Source: By "Junkyardsparkle" - Own work, CC0.: <https://commons.wikimedia.org/w/index.php?curid=42604043>

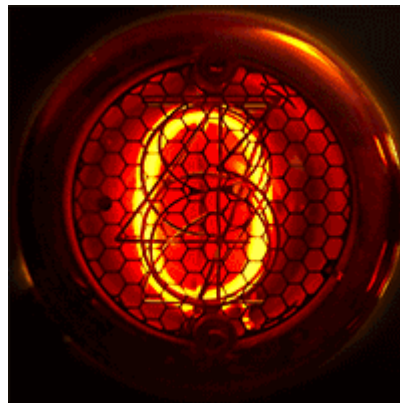


*Fig.5.7.4. A neon lamp with figures, a so-called Nixie tube.
Ref. <https://upload.wikimedia.org/wikipedia/commons/4/4f/Nixie2.gif>*

Tubes with low pressure mercury

Tubes emitting UV light

A small amount of mercury (Hg) is added to an evacuated quartz tube with an electrode at each end of the tube. Some of the mercury is evaporated and exists as a gas phase in the tube. The gas in the tube is ionized by using a relatively high potential difference (could be about 1 kV) between the electrode at the one end and the electrode at the other end. The high potential difference starts the plasma glow process in the tube. Once started, the plasma continues to glow at a much lower potential difference. Electrons are emitted from the cathode. Both electrodes are heated, thereby it is possible to keep the potential difference between the two

electrodes at a much lower value than with cold electrodes. Hg atoms are excited by collisions with electrons in the plasma. When the excited Hg atoms return to the ground state (the not excited state), UV light is emitted from the Hg atoms in the plasma and emitted from the quartz tube. Such an UV light may be used to produce ozone and kill bacteria in food industries. The tubes with uv light are also used in solariums. The tube is made of quartz in order to allow the UV radiation to emerge.



Fig.5.7.5. Low-pressure mercury-vapor glow discharge.

By Deglr6328 - Own work, CC BY-SA 3.0,

<https://commons.wikimedia.org/w/index.php?curid=1580558>

Tubes emitting visible light



Fig.5.7.6 Top: two compact fluorescent lamps. Bottom: two fluorescent tube lamps.

Ref. Christian Taube, CC BY-SA 2.0 de

<https://commons.wikimedia.org/w/index.php?curid=102772>

This type of tube is constructed in the same way as described above for the UV light tube. The only difference is, that the inside of the tube is covered by a fluorescent coating. This coating absorbs the UV light coming from the mercury plasma in the tube. Atoms in the coating are excited when they absorb UV light. Electrons in the atoms jump to a higher energy level in one step. Right after electrons return to the original

energy level in more steps. Hence, a smaller amount of energy is released in each of the smaller steps. The corresponding wavelength is larger than it is for the UV light. It is in the visible spectrum. Different atoms in the coating may emit light with different wavelength and different colours. In this way one may produce *fluorescent tubes with different colours*. If different colours are combined, it is possible to produce *fluorescent tubes with white light*. Fluorescent tubes with the highest quality white light combine 5 different colours in the fluorescent coating. The tube itself is made of glass, not quartz. Glass is much cheaper than quartz and glass is lesser transparent to UV light. We don't want the white light sources to emit UV light.

Tubes generating white light have been used widely for decades everywhere at homes, at factories and as street light. The tubes are still in common use. However, the LED light sources have conquered some part of the market for the tubes. The LED light sources are only slightly more efficient than the fluorescent tubes.



Fig.5.7.7 High pressure mercury 125 W lamp

By Kukdide - Own work, CC BY-SA 3.0

<https://commons.wikimedia.org/w/index.php?curid=7686237>

5.8 Low pressure mercury tubes in more detail

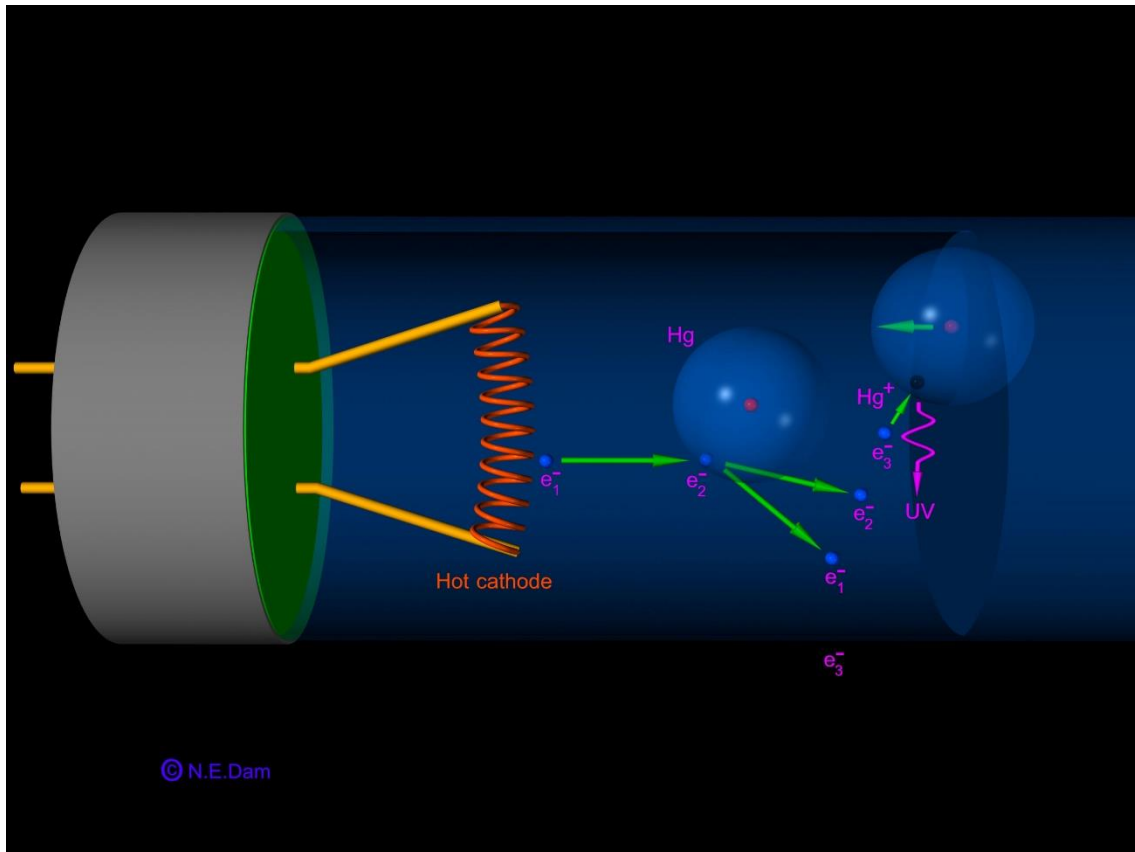


Fig.5.8.1, ref.1 (N.E. Dam ©). Left end of a low pressure mercury tube with a hot cathode.

Fig.5.8.1 shows the hot cathode in a low pressure mercury tube. It could be an UV or visible light tube. The fluorescent coating – if there is one - is not shown here.

There is a current flow from the one (yellow) terminal through the (orange) filament wire to the other (yellow) terminal. It keeps the filament wire at a relatively high temperature allowing electrons to “evaporate” from the surface of the filament wire.

Fig.5.8.2 shows the anode. It has a high positive potential relative to the cathode. When starting the plasma glow process, the potential difference may be several hundreds of volt. This potential difference is enough to cause electrons from cathode moving away from the cathode in direction of the anode.

Fig.5.8.1 shows one of these electrons (e_1) colliding with a mercury (Hg) atom. The electron e_1 causes electron e_2 to be excited away from the Hg atom. Now, there are two free electrons, e_1 and e_2 . They are both able to excite other Hg atoms, thereby starting an avalanche effect propagating through the entire tube to the other of the tube, where the anode absorbs many of the electrons. Once the plasma glow process has started, the potential difference between the two electrodes may be reduced to about 230 V. UV light is emitted when one of the electrons in the plasma (e_3 in the figure) is attracted to one of the Hg^+ ions, which is recombined to a neutral Hg atom.

Fig.5.8.3 shows the entire tube.

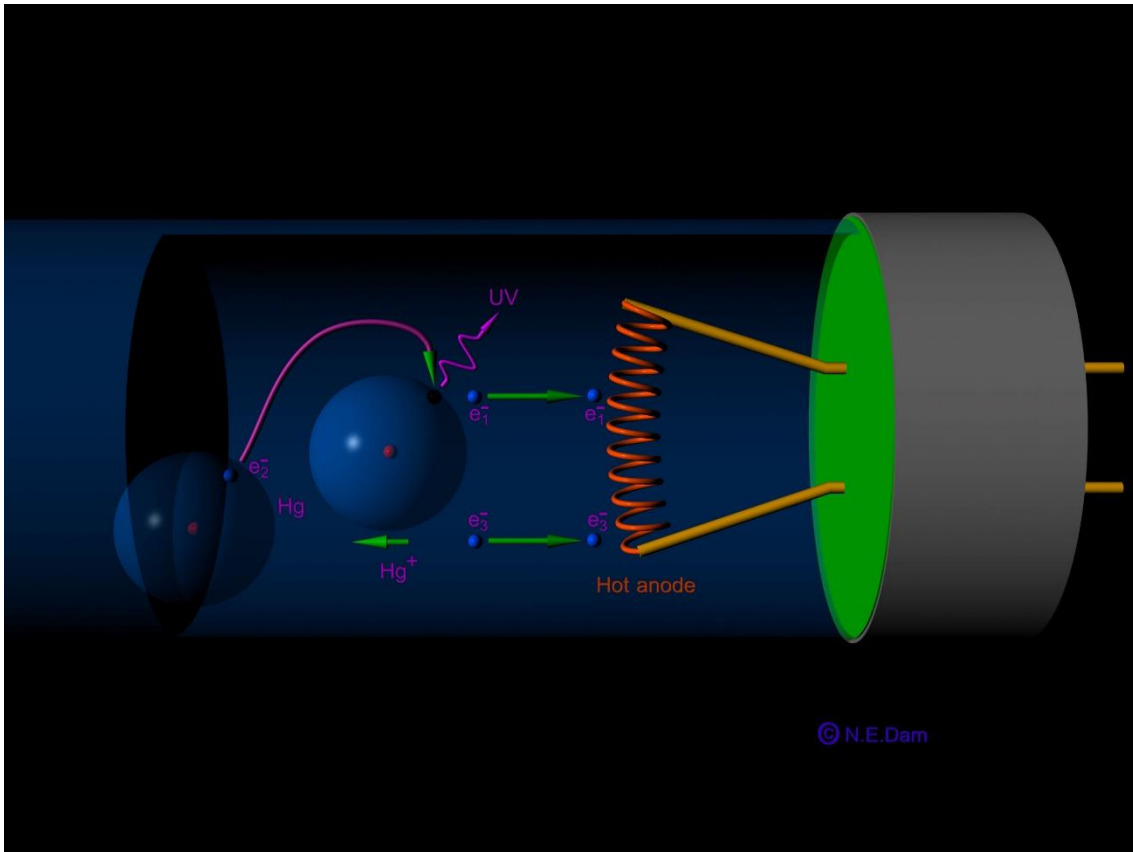


Fig.5.8.2, ref.1 (N.E. Dam ©). Right end of a low pressure mercury tube with a hot anode.

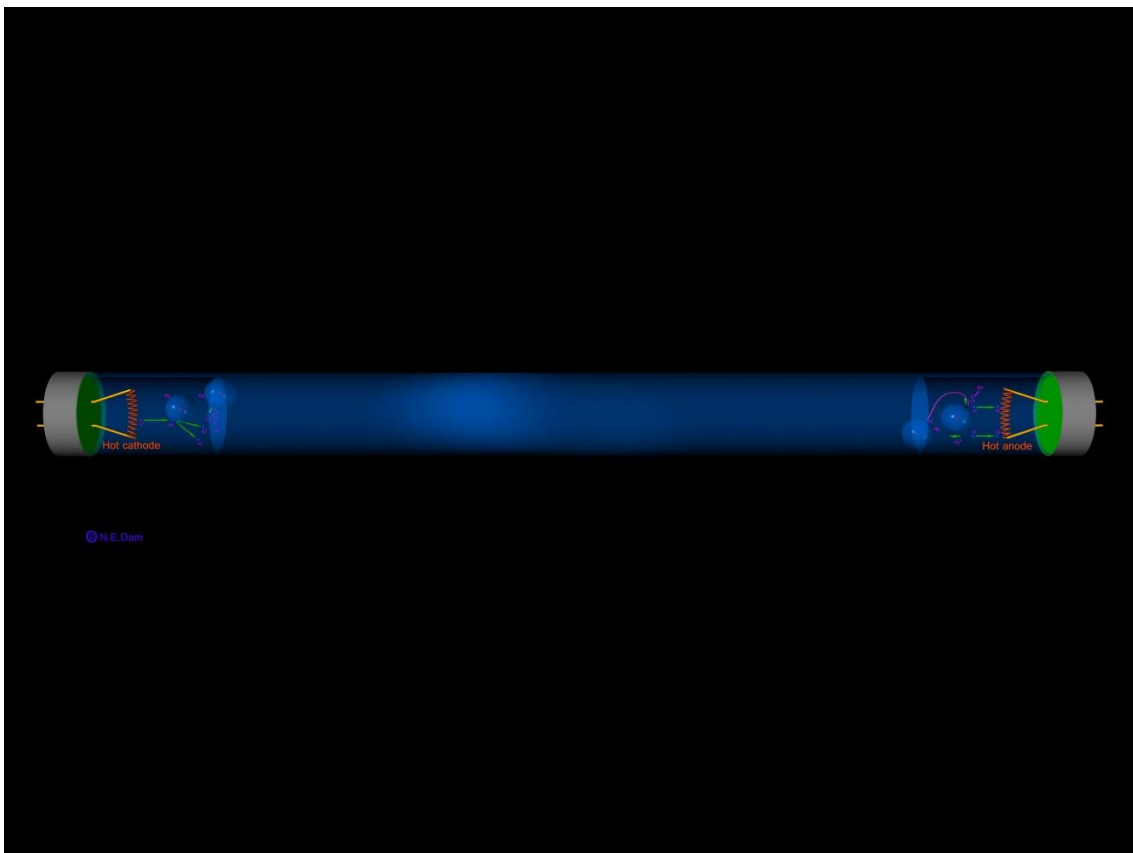


Fig.5.8.3, ref.1 (N.E. Dam ©). The entire low pressure mercury tube with hot electrodes.

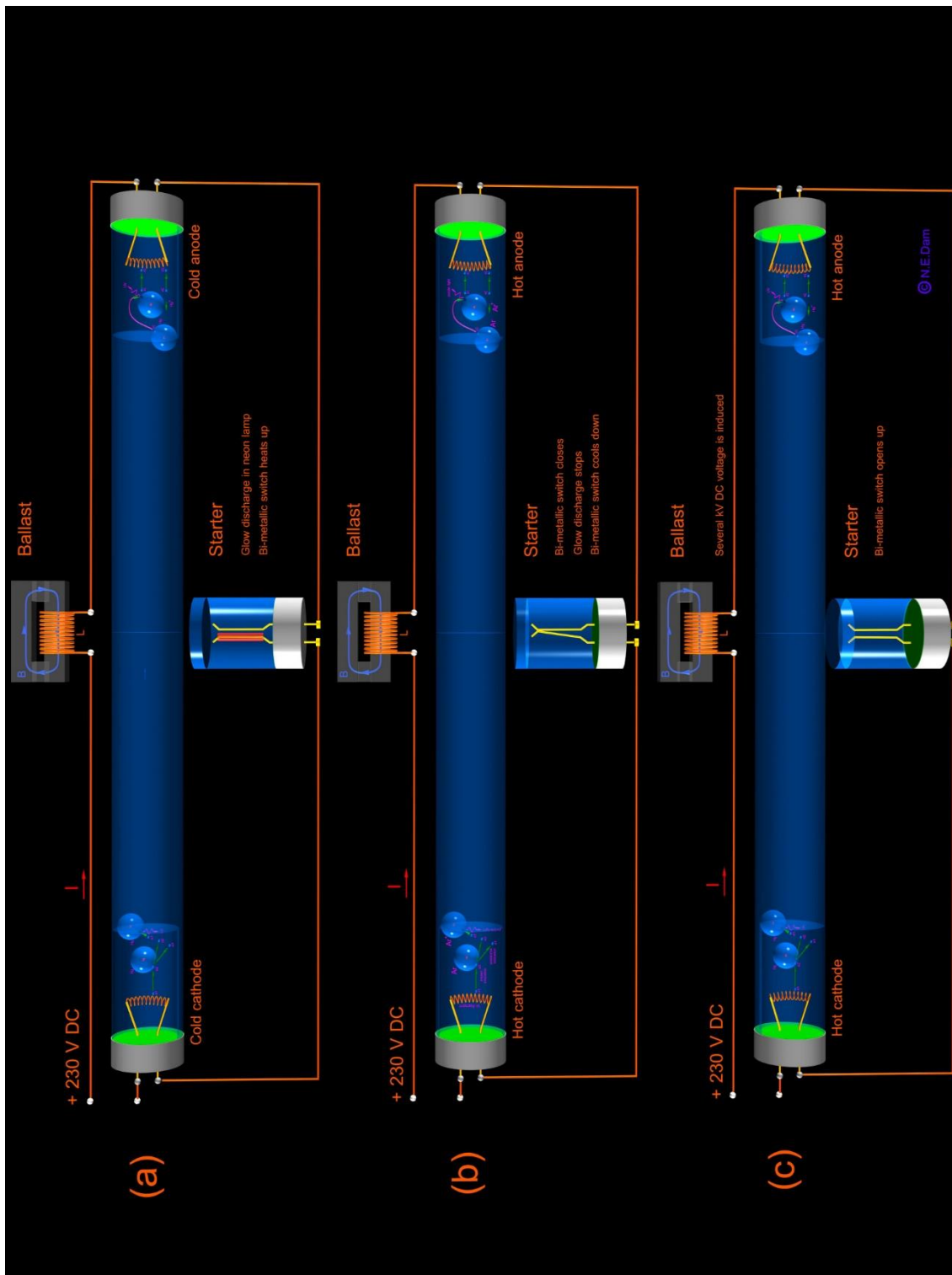


Fig.5.8.4, ref.1 (N.E. Dam ©). The start-up process of a low pressure mercury tube.

Fig.5.8.4 shows the start-up process with a DC supply voltage of 230 V:

- 230 V DC is connected to the + marked terminal and 0 V to the other input terminal. The +230 V DC is connected through the coil in the self inductance L, through the cold anode filament to the anode in the neon lamp in the so-called starter. The 0 V is connected through the cold cathode filament to the cathode in the neon lamp. In other words the 230 V DC supply is connected to the neon

lamp in the starter. Now, the cathode in the neon lamp emits electrons starting a plasma glow process in the neon lamp. The 230 V is enough to start the glow process in the relative small neon lamp. The figure shows the glowing plasma around the cathode, which is heated this way. This causes the bi-metallic cathode to bend towards the anode making a short circuit in the neon lamp, see the next figure.

- b. The short circuit in the neon lamp causes a current I to flow from the + terminal, through the self inductance L , through the cold anode filament, through the neon lamp (the starter) and finally through the cold cathode to the 0 V terminal. The current I heats up the filaments in two electrodes. At the same time, the neon lamp has stopped glowing because of the short circuit in the neon lamp. The cathode in the neon lamp cools down and the bi-metal in the cathode straight out again, see the next figure.
- c. When the short circuit in the neon lamp brakes, it brakes the current flow through the entire circuit from the + terminal to the 0 terminal. This causes the self inductance to induce a very high voltage at the anode and this starts the plasma glow process in the tube as described in the beginning of this section. When the plasma glow process has started in the tube, the resistance of the plasma decreases very much and the current flows through the plasma, not through the neon lamp, which is inactive from now on.

The above description was based on a 230 V DC supply voltage. However, it also works with 230 V 50 Hz AC supply voltage. In this case, the cathode and anode swaps 100 times per second meaning that the cathode is 50 times at the one end every second. If operated with AC, a capacitor is added to the circuit in order to compensate for the self inductance and reduce the phase shift between the voltage and the current. It may be important if there are many tubes in action at the same time.

Nowadays, electronic equipment is used to start the plasma glow in the tubes. It reduces the wear on the tubes and increases the efficiency.

A detail, which has not been mentioned so far is that there are also some Ar atoms in the gas in the tube, not only Hg atoms. The plasma glow process described above is in fact started with Ar atoms causing a purple light. Later on, when the Ar plasma has heated up the gas in the tube, a larger part of the Hg atoms is evaporated to the gas phase and the Hg(g) atoms takes over the glow process emitting UV light.

5.9 Equipotential curves and surfaces

Electric fields may be visualized by E-field lines.

Electric potentials may be visualized by equipotential curves.

The equipotential curves are used for 2D illustrations of the electric potential.

An equipotential curve is a curve through points of the same electric potential. The E-field is perpendicular to the curve at all points on the curve.

Equipotential curves

Fig.5.9.1 shows some equipotential curves around the two charges $+q$ and $-q$ in a dipole. The charges $+q$ and $-q$ may reside on the outer surface of two conducting spheres. All points on the curve marked V_4 have the same electric potential $V=V_4$ and likewise for V_1 , V_2 , and V_3 . Note that the orange E-field lines are perpendicular to the green electric potential curves at all points.

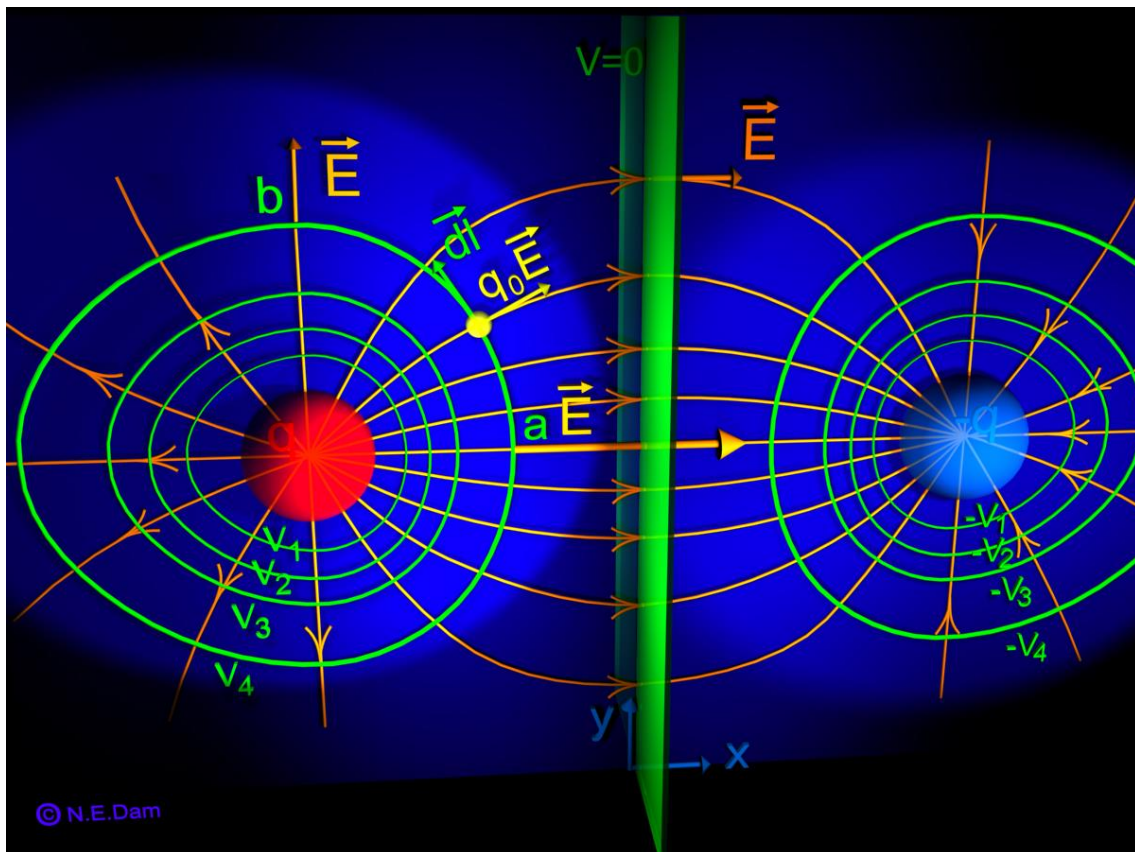


Fig.5.9.1, ref.1 (N.E. Dam ©). Equipotential curves around the two charges in a dipole.

If a charge q_0 is moved from a to b along the equipotential curve V_4 , the E-field $\vec{E}$ and the electric force $\vec{F}_E$ are both perpendicular to the path $\vec{dl}$ at all points along the curve. Therefore

$$W_{ab} = \int_a^b \vec{F}_E \cdot \vec{dl} = q_0 \int_a^b \vec{E} \cdot \vec{dl} = 0$$

Hence, the electric force does no work on q_0 when q_0 is moving along an equipotential curve. The work W_{ab} is also zero if q_0 is orbiting around $+q$ like an electron orbiting around an atomic nucleus.

Now, let us consider the positive charge $+q$ in the left part of the figure. As we remember from fig.5.3.3, the electric potential falls off most rapidly close to the charge. Therefore the equidistant levels $V_1, V_2, V_3, V_4, \dots$ are relatively closely spaced close to the charge $+q$.

Fig.5.9.1 shows, that there is a plane in the middle between $+q$ and $-q$ where the electric potential is zero. It is the green plane marked $V=0$, which is perpendicular to the direct connection line between $+q$ and $-q$. The E-field lines and the equipotential curves in this figure only represent the E-field and the electric potential in the blue xy-plane. This plane includes the two charges $+q$ and $-q$ and it is perpendicular to the green $V=0$ plane. Therefore, it is only a 2D representation.

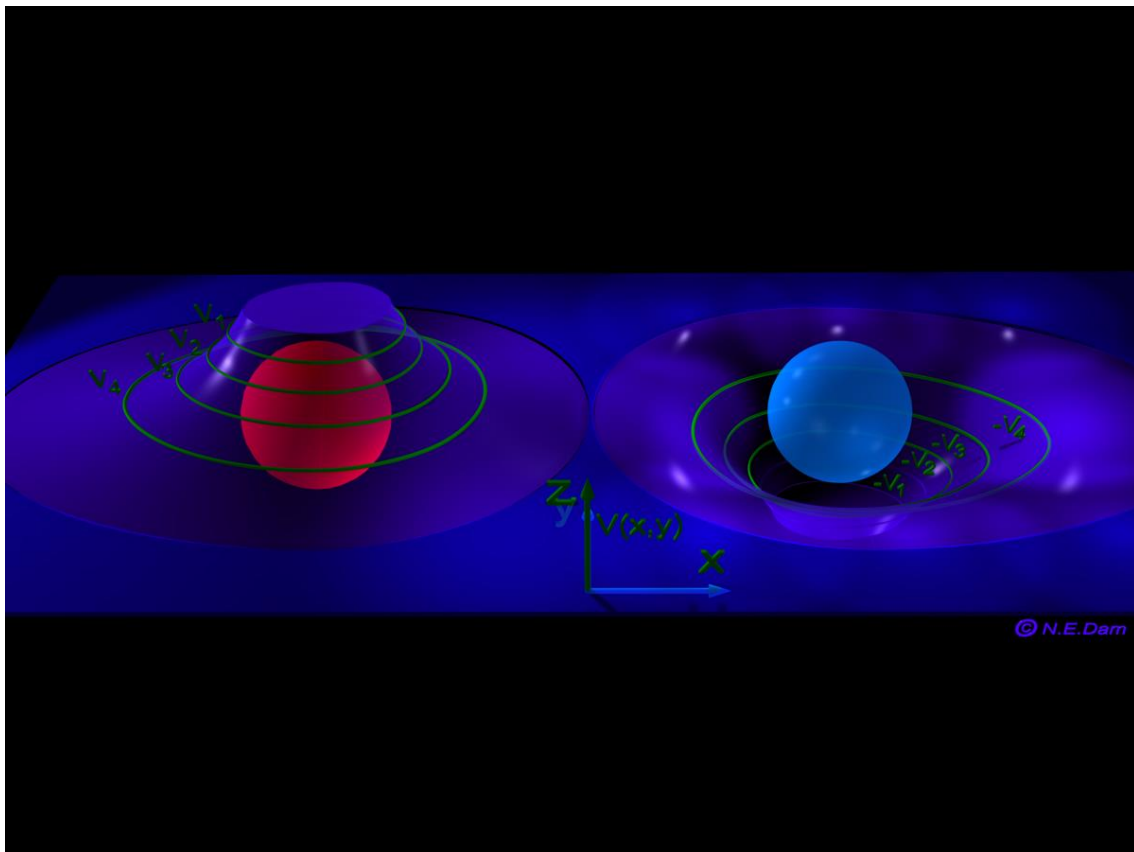


Fig.5.9.2, ref.1 (N.E. Dam ©). A 3D representation of the 2D equipotential curves around the two charges in a dipole.

Fig.5.9.2 shows a depiction, where the height of the purple surface relative to the blue flat xy-plane is $z=V(x,y)$. If we consider a point (x,y) in the horizontal xy-plane, we have the electric potential $V(x,y)$ at that point. Then we plot a point (x,y,z) above the point (x,y) in such a way, that the vertical distance z from the blue xy-plane (where $z=0$) represents the electric potential $V(x,y)$ at that point (x,y) . All the plotted points creates the purple surface like a landscape. We have a high hill with a flat top in the

purple landscape where the electric potential is high as it is through the conducting sphere of charge $+q$.

Likewise we have a depression with a flat minimum, where the electric potential is highly negative as it is through the conducting sphere of charge $-q$.

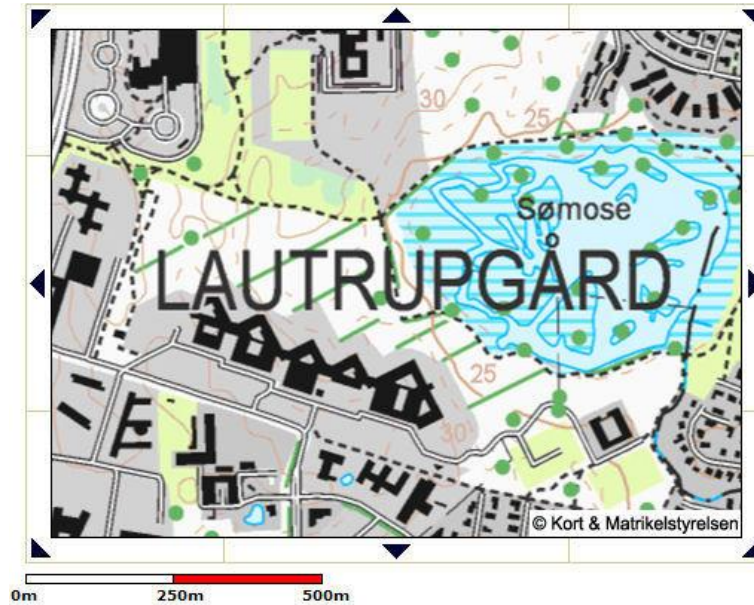


Fig.5.9.3, Ref. The National Survey and Cadastre is Denmark's public authority for maps and geodata, see www.kms.dk.

Please note, that the purple surface is not an equipotential surface. We have many different electric potentials on the purple surface. The purple surface is a 3D visualization of the electric potential $V(x,y)$ in a 2D xy -plane.

This is analogous to a topographic map, see fig.5.9.3. A topographic map shows the levels in the landscape by using curves through the same level. We see a curve marked by 25 around the lake "Sømosse". It is the locus for points of level 25 m above the sea level. The sea level is not the level of the lake in the figure, which is about 25 m. The sea level is a reference, which is the international average level of the ocean. Curves are drawn through points at the same level. If the curves are close together, the landscape is steep. If the curves are spaced out, the landscape is rather flat.

It is possible to make a 3D model of the landscape from a topographic map if we create a surface, where the height h of the surface (relative to a horizontal reference plane at $h=0$) agree with the height curves in the map. We should do that, if we want to create a model railway landscape, that should be a copy of a real landscape at some selected location.

Equipotential surfaces

The electric potential is illustrated in 3D space using equipotential surfaces.

An equipotential surface is a surface through points of the same electric potential. The E -field is perpendicular to the surface at all points on the surface.

If a test charge q_0 moves around on an equipotential surface, the E-field $\vec{E}$ and the electric force $\vec{F}_E$ are both perpendicular to the movement $d\vec{l}$ of the charge. Therefore the electric force does no work on the charge as mentioned in the above section with the equipotential curves. The electric potential energy $U_{\text{pot}} = q_0V$ and the electric potential V remain constant, when the charge moves around on the surface.

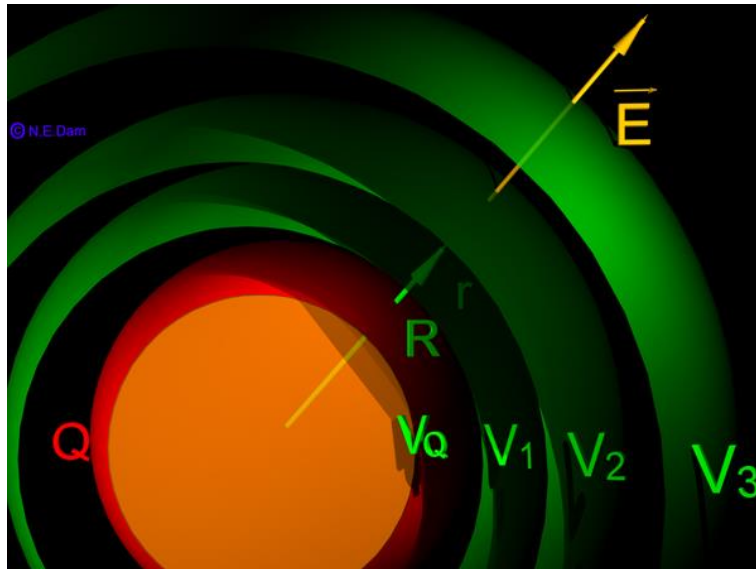


Fig.5.9.4,ref.1 (N.E. Dam ©). Equipotential surfaces around a conducting sphere.

The figures below show some examples of equipotential surfaces around charged, conducting spheres.

Fig.5.9.4 shows some equipotential surfaces around a conducting sphere. On the surface of the sphere, the electric potential is

$$V_Q = V_R = \frac{q}{4\pi\epsilon_0} \frac{1}{R} \quad \text{eq. 5.3.3}$$

Fig.5.3.3 shows, that the electric potential drops rather steeply just outside the surface and then more slowly at a further distance from the sphere. If the electric potential levels $V_1, V_2, V_3, \dots$ are spaced equidistantly along the vertical electric potential axis in fig.5.3.3, then the equipotential surfaces are relatively closely spaced along the horizontal r -axis just outside the surface.

Note, that the equipotential surfaces are dense, where the E-field lines are dense. E.g. if a positive test charge q_0 is moved from point c towards point d across the dense equipotential surfaces, the electric force performs a large amount of work on the test charge over a short distance while the electric potential drops fast and the kinetic energy increases fast giving the test charge a great acceleration. This is analogous to the situation, where the gravity force pulls a mass down from a steep hill.

Fig.5.9.5 shows some equipotential surfaces for an electric dipole and some E-field lines.

Note, that the field magnitude (or the field line density) may not be the same everywhere on an equipotential surface. E.g. we have a high field line density and hence a strong E-field at point a. At point b, which is on the same equipotential surface (V_4), we have a lower field line density, and hence a weaker E-field.

Note also, that all field lines are perpendicular to the equipotential surfaces.

The green vertical plane is the equipotential surface for $V=V_0=0$.

The equipotential surfaces are more densely packed on the right side of $+q$ (between the $+q$ and the green vertical plane marked V_0) than on the left side of $+q$. It is because the electric potential drops from the maximum value V_Q to zero over a shorter distance on the right side of $+q$ than on the left side of $+q$.

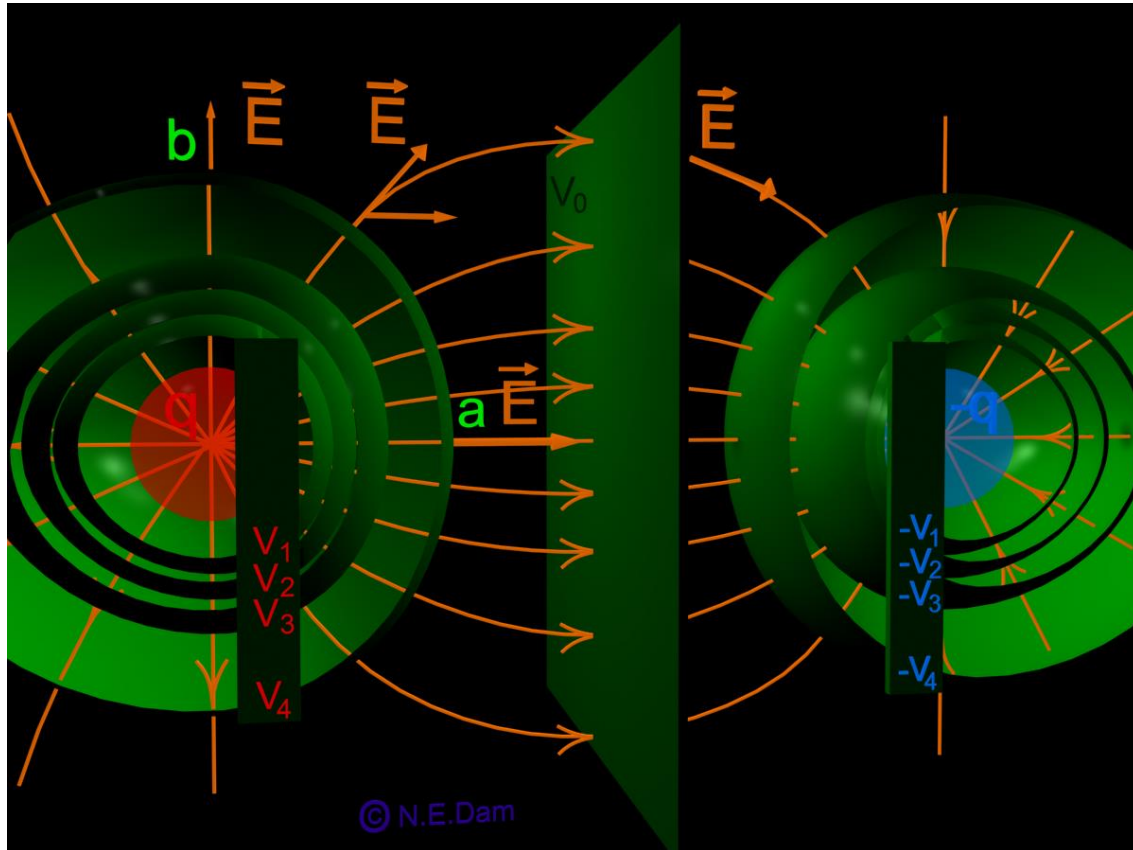


Fig.5.9.5.ref.1 (N.E. Dam ©). Equipotential surfaces for a dipole. The equipotential surfaces are cut over to see the charges inside the surfaces. Besides, the two vertical green plates are inserted as a visual background for V_1, V_2, V_3, V_4 and $-V_1, -V_2, -V_3, -V_4$.

The E-field is perpendicular to a conducting surface

Fig.5.9.6 shows the field pattern and a cut through one of the electric potential surfaces between a positive charge and a flat conducting surface. Since the conducting surface is an equipotential surface, the field lines are perpendicular to the conducting surface. It will be proved in below.

Fig.5.9.7 shows how the field pattern may be constructed in a simple case. If we imagine, that the conductor is removed and replaced by a negative charge placed symmetrically on the right side of the conducting surface, then we see how the E-field pattern may be.

Now we want to prove, that the E-field lines are perpendicular to a flat conducting surface, see fig.5.9.8.

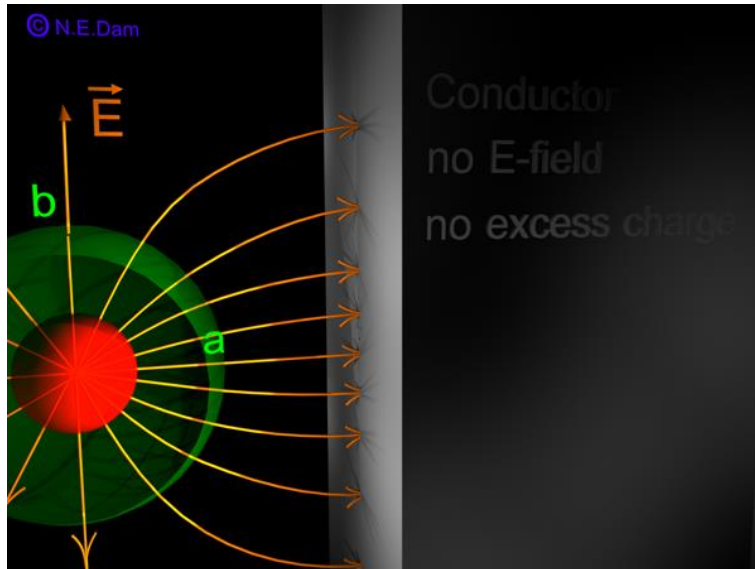


Fig.5.9.6, ref.1 (N.E. Dam ©). The field pattern and an electric potential surface between a charge and a conducting surface.

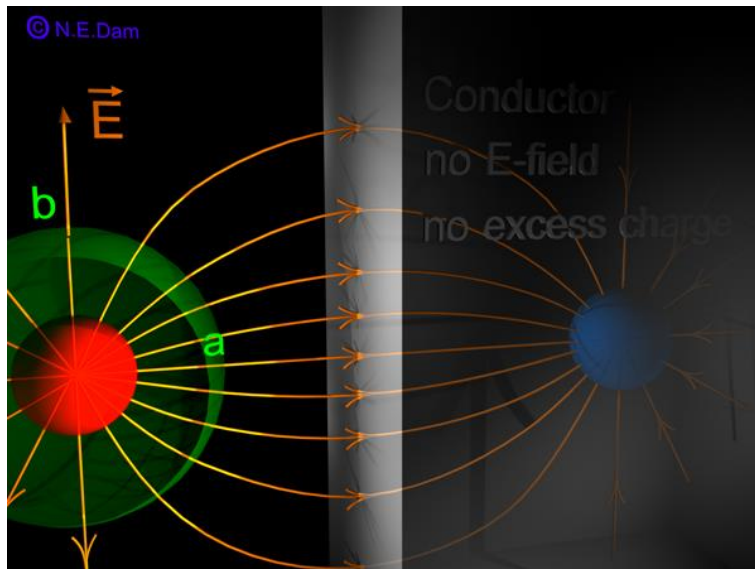


Fig.5.9.7, ref.1 (N.E. Dam ©). The field pattern is the same as if there is a fictitious (negative) mirror of the positive charge inside the conductor. Of course there is no such charge inside the conductor, but the picture is useful.

We use eq.5.1.5 along the closed integration path a-b-c-d-a.

$$\oint \vec{E} \cdot d\vec{l} = 0 \quad \text{eq.5.9.1}$$

The integral is spilt up into four integrals:

$$\oint \vec{E} \cdot d\vec{l} = \int_a^b \vec{E} \cdot d\vec{l} + \int_b^c \vec{E} \cdot d\vec{l} + \int_c^d \vec{E} \cdot d\vec{l} + \int_d^a \vec{E} \cdot d\vec{l} = 0 \quad \text{eq.5.9.2}$$

Since all excess charges are at rest, the E-field is zero inside the conductor. Therefore, the integral inside the conductor (from d to a) is zero

$$\int_d^a \vec{E} \cdot d\vec{l} = 0 \quad \text{eq. 5.9.3}$$

For the integral of $\vec{E} \cdot d\vec{l}$ along the two paths from a to b and from d to c we have

$$\int_a^b \vec{E} \cdot d\vec{l} \cong \int_d^c \vec{E} \cdot d\vec{l} \quad \text{eq. 5.9.4}$$

This is because the E-field along the path a-b is approximately the same as the E-field along the path d-c. Therefore, we get

$$\int_a^b \vec{E} \cdot d\vec{l} \cong - \int_c^d \vec{E} \cdot d\vec{l} \quad \text{eq. 5.9.5}$$

The only integral left in the sum of the four integrals eq.5.9.2 is the integral of $\vec{E} \cdot d\vec{l}$ from b to c, which is parallel with the surface. According to eq.5.9.2 this integral should be zero

$$\int_b^c \vec{E} \cdot d\vec{l} = 0 \quad \text{eq. 5.9.6}$$

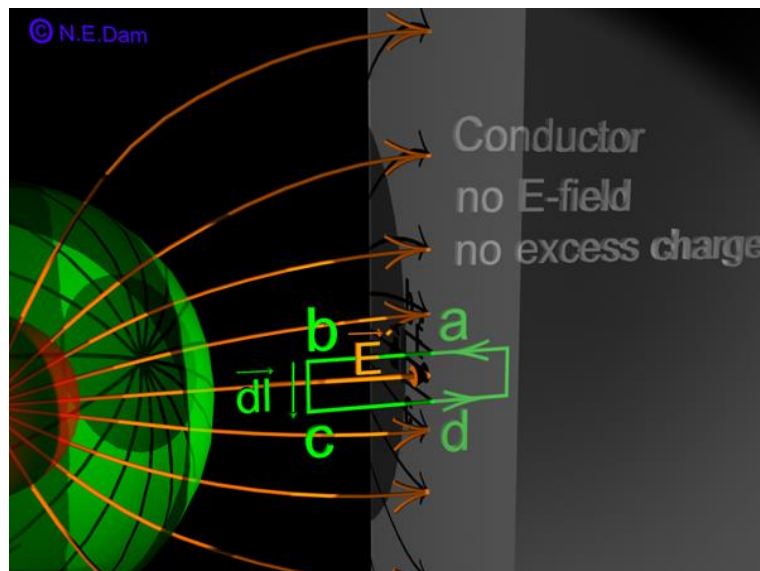


Fig.5.9.8, ref.1 (N.E. Dam ©). The E-field pattern and a cut through one of the equipotential surfaces between a positive charge and a conducting surface. The positive charge and the field lines cast shadow on the inner side of the equipotential surface to give 3D effect.

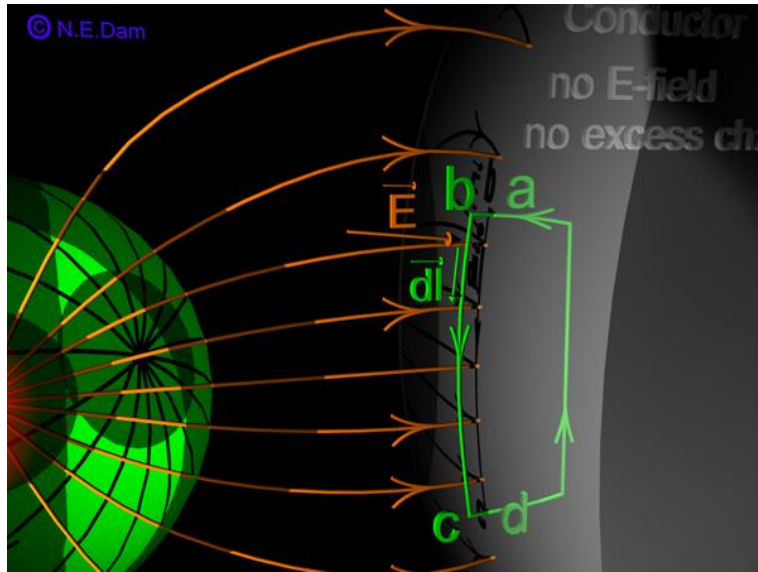


Fig.5.9.9, ref.1 (N.E. Dam ©). Same figure as above, but with a curved conducting surface. The closed path from b to c is in parallel with the conducting surface.

The integral is zero if

$$\vec{E} \cdot d\vec{l} = 0 \quad \text{eq. 5.9.7}$$

which means that

$$\vec{E} \perp d\vec{l} \quad \text{eq. 5.9.8}$$

Since $d\vec{l}$ is in parallel with the surface, we have that $\vec{E}$ must be perpendicular to the surface.

In case where the surface is curved like in [fig.5.9.9](#) we have exact the same proof. Also here $d\vec{l}$ is in parallel with the surface at all points along the integration path from b to c.

Hence $\vec{E}$ must be perpendicular to the surface no matter if the surface is flat or curved.

The electric potential is constant inside a cavity in a conductor

Now, we will prove, that we have the same electric potential inside a cavity.
Fig.5.9.10 shows a cavity in a conductor. There is no charge inside the cavity.

We are going to prove, that the (violet) equipotential surface V_a inside the cavity has the same electric potential as the inner surface of the cavity, which is V_b , i.e

$$V_a = V_b \text{ eq.5.9.9}$$

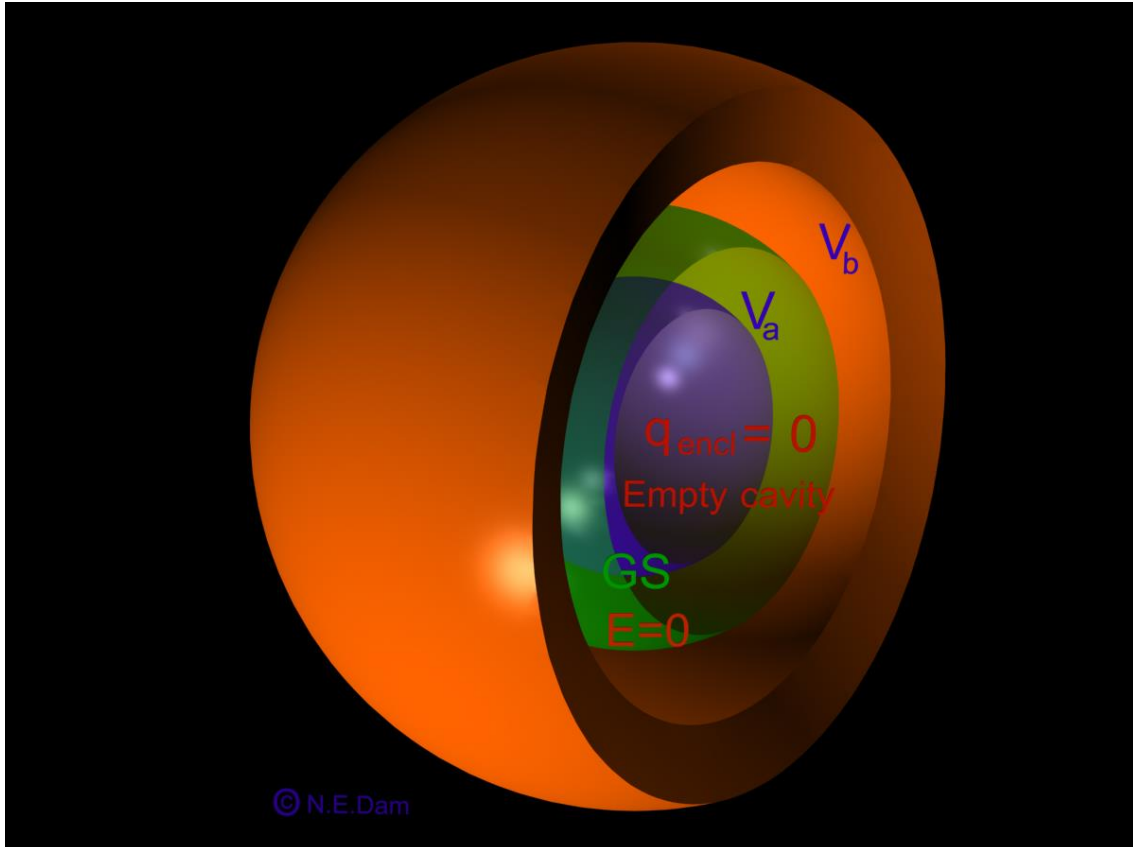


Fig.5.9.10,ref.1 (N.E. Dam ©). The proof for the electric potential is constant inside a cavity in a conductor. The cavity, the green Gaussian surface and the purple equipotential surface V_a are cut through.

We place a (green) Gaussian surface between the equipotential surface V_a and the electric potential surface V_b . Since the enclosed charge is zero, the E-field at the (green) Gaussian surface will also be zero according to Gauss's law. Then eq.5.1.4 gives

$$\int_a^b \vec{E} \cdot d\vec{l} = V_a - V_b = 0 \Rightarrow V_a = V_b \text{ eq.5.9.10}$$

5.10 The electric potential gradient

General properties of the electric potential gradient

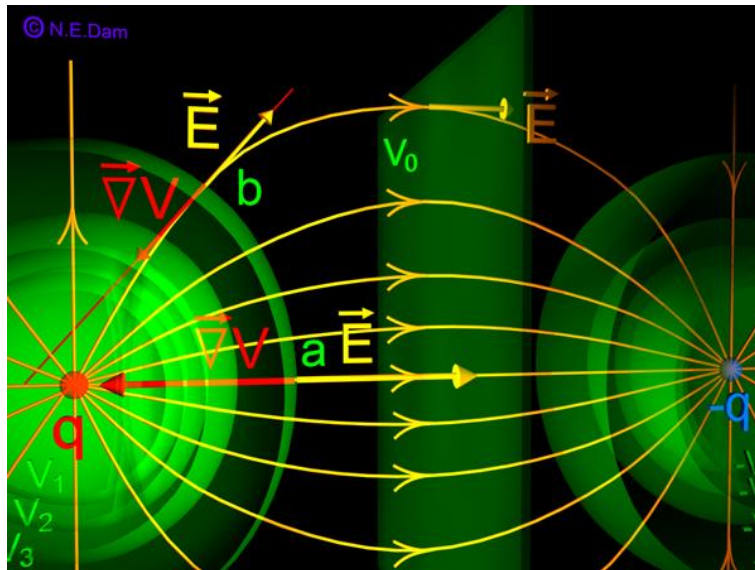


Fig.5.10.1,ref.1 (N.E. Dam ©). The red gradient vector points in the direction where the electric potential increases most rapidly. This direction is perpendicular to the equipotential surfaces. The direction is not always the same as the direction directly to the center of the positive charge.

Fig.5.10.1 shows a red vector $\vec{\nabla}V$ beside the orange $\vec{E}$ vector at points a and b. In the next section we will prove that

$$\vec{E} = -\vec{\nabla}V \quad \text{eq. 5.10.1}$$

The red vector $\vec{\nabla}V$ is the so-called gradient vector of the electric potential. The symbol ∇ is called “Nabla”. It is a mathematical sign, not a Greek letter.

The figure shows, that $\vec{\nabla}V$ and $\vec{E}$ have the same magnitude. However, they are in opposite directions.

The E-field line from q to point a is a straight line. Hence the tangent of the E-field line at point a goes through q. It means that $\vec{\nabla}V$ and $\vec{E}$ are pointing directly towards q respectively, directly away from q.

At point a we have, that $\vec{\nabla}V$ is pointing directly towards q where we have the highest electric potential.

The E-field line from q to point b is not a straight line. In this case the tangent of the E-field line at point b does not pass through q as shown in the figure. However, the tangent is perpendicular to the equipotential surface at point b. Therefore, at point b we have, that the gradient $\vec{\nabla}V$ does not point directly towards q, where we have the highest electric potential. Instead, the gradient $\vec{\nabla}V$ is pointing towards the next equipotential surface (V_3).

At any point (x,y,z) we have that the gradient $\vec{\nabla}V$ points in the direction, where the electric potential V increases most rapidly. This direction is perpendicular to the equipotential surface at point (x,y,z) .

Why are we so interested in the gradient of the electric potential V ?

The reason is the following.

Usually, it is more easy to calculate the scalar electric potential $V(x,y,z)$ rather than the vector field $\vec{E}(E_x(x,y,z), E_y(x,y,z), E_z(x,y,z))$.

If we want to calculate the electric potential $V(x,y,z)$ at a point (x,y,z) , we should only sum the *scalar* electric potentials $V_1, V_2, V_3, \dots$ from the different charges $q_1, q_2, q_3, \dots$

If we want to calculate the vector field $\vec{E}(E_x(x,y,z), E_y(x,y,z), E_z(x,y,z))$ at a point (x,y,z) , we should sum the *vectors* $\vec{E}_1, \vec{E}_2, \vec{E}_3, \dots$ from the different charges $q_1, q_2, q_3, \dots$ as we learned in chapter 2. It is more complicated to sum the 3 coordinates in the vectors than it is to sum scalars. Besides, all 3 coordinates in the E-field vector $\vec{E}$ are functions of (x,y,z) .

When we have found the scalar electric potential $V(x,y,z)$, we can use eq.5.10.1 to calculate the E-field vector at all points (x,y,z) .

When we calculate the E-field using eq.5.10.1 we have to differentiate V with respect to x, y and z as we will see later. Hence we should know V as a function of x, y and z . This little complication means that if we have only a small number of charges, it is easier to sum the vectors $\vec{E}_1, \vec{E}_2, \vec{E}_3, \dots$ if we want the total $\vec{E}$. However, if we have many charges and we should find the total $\vec{E}$, it is usually more easy to sum the electric potentials $V_1, V_2, V_3, \dots$ from the different charges $q_1, q_2, q_3, \dots$ and then differentiate the resulting $V(x,y,z)$ to get the E-field.

In order to summarize, we have two methods to calculate an E-field:

1. Adding the E fields from all the point charges (chapter 2)
2. Finding the electric potential V from all the point charges and then differentiating V to get the E-field.

Method 1 is more complicated, since we have to sum the 3 components in $\vec{E} = (E_x, E_y, E_z)$ from each charge element separately, and each of the components is a function of x, y and z .

Method 2 is easier if we have many charges, since we only have to sum or integrate a scalar function $V(x,y,z)$ and then differentiate the resulting V to get the field.

If we know E we can use eq.5.1.4 to find V :

$$V_a - V_b = \int_a^b \vec{E} \cdot d\vec{l} \quad \text{eq. 5.10.2}$$

Calculation of an increase in electric potential V

This section should give the reader an understanding of the concept of the gradient and the electric potential function. As an illustration of the theory we use the electric potential function of two variables $V(x,y)$ shown in fig.5.9.2. A close-up of the left part of this figure is shown in fig.5.10.2. We want to examine how $V(x,y)$ changes, when (x,y) changes from (x_0, y_0) to $(x_0 + dx, y_0 + dy)$. The change in coordinates may be described by the vector $\vec{dl}$ from (x_0, y_0) to $(x_0 + dx, y_0 + dy)$. When we “move” along $\vec{dl}$ from (x_0, y_0) to $(x_0 + dx, y_0 + dy)$ the electric potential increases by the amount

$$\Delta V = V(x_0 + dx, y_0 + dy) - V(x_0, y_0) \quad \text{eq.5.10.3}$$

The following derivation will show how to find dV

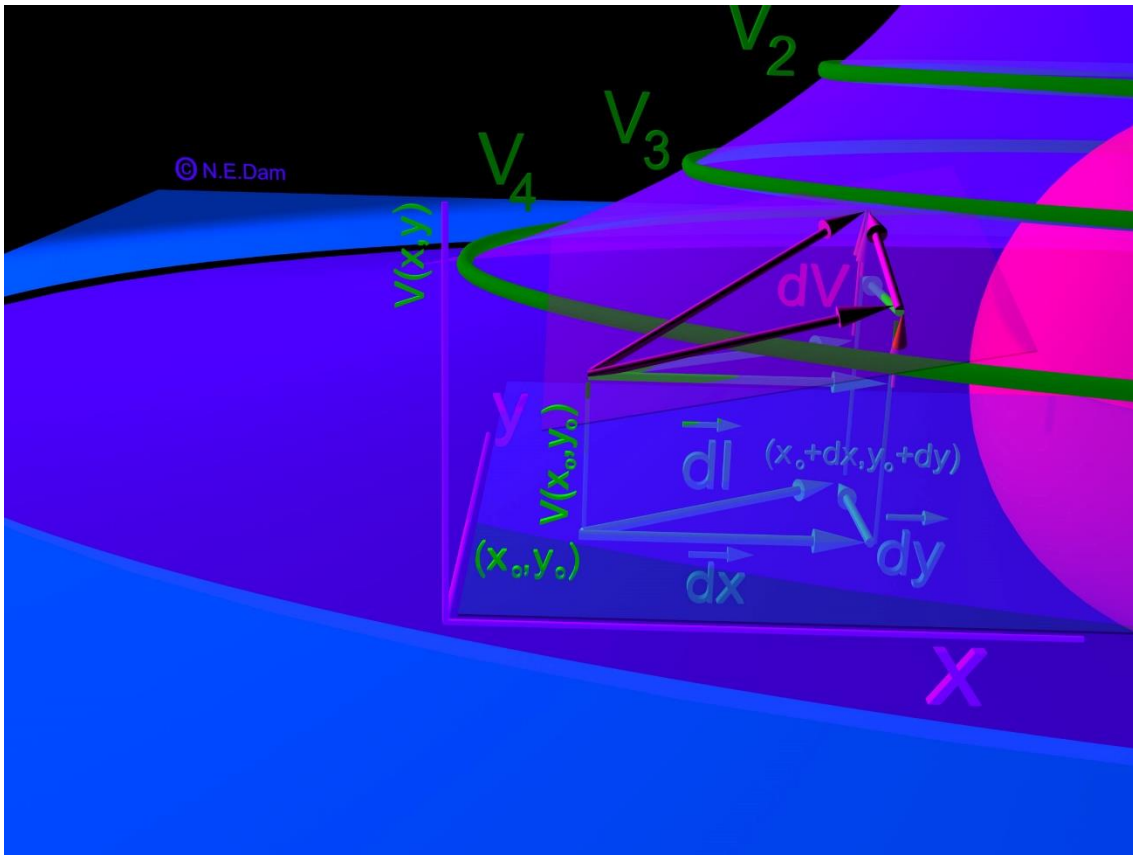


Fig.5.10.2.ref.1 (N.E. Dam ©). This figure is a close-up of the left part of fig.5.6.2. We will examine how $V(x,y)$ changes, when (x,y) changes from (x_0,y_0) to $(x_0 + dx, y_0 + dy)$.

The so-called *total differential function* dV is an approximation to ΔV . The increase dV in V is calculated using partial differential coefficients at point (x_0, y_0) .

Vector $\vec{dl}$ may be considered as a sum of the two vectors $\vec{dx}$ and $\vec{dy}$.

Instead of moving from (x_0, y_0) to $(x_0 + dx, y_0 + dy)$ along $\vec{dl}$ we may also move along $\vec{dx}$ and thereafter along $\vec{dy}$. When we move along $\vec{dx}$ parallel with the x-axis, we have that y must be constant, and $y = y_0$ defines the vertical blue plane shown in fig.5.10.3. For all points on this plane we have the same value for y which is y_0 . When we move along $\vec{dx}$ we have that the value of the electric potential should be on the purple

electric potential surface and, at the same time, also on the vertical plane $y = y_0$. The electric potential will therefore follow the intersection curve between the purple surface for $V(x,y)$ and the vertical blue plane given by $y = y_0$. This intersection curve is $V(x)$. When y is constant, the electric potential only depends on x . The slope of the curve $V(x)$ is written the following way:

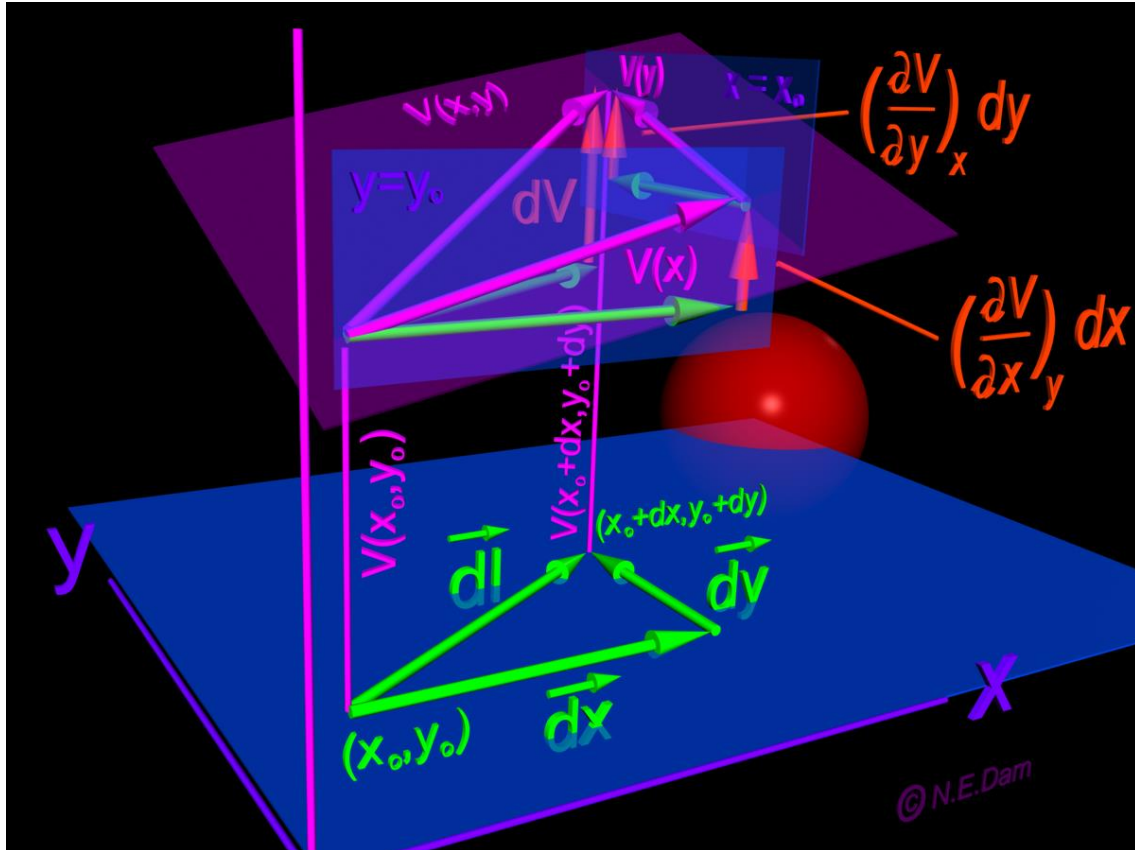


Fig.5.10.3, ref.1 (N.E. Dam ©). The red charge gives an electric potential distribution in the blue (x,y) plane. At any point (x,y) we have a value of the electric potential $V(x,y)$. This value is used to construct a point on the purple surface where the first and second coordinates are (x,y) and the third coordinate is $V(x,y)$. The purple surface shows thereby the electric potential V as a function of (x,y) . The purple surface in this figure is the same as the purple surface in fig.5.10.2..

$$\left(\frac{\partial V}{\partial x}\right)_y \quad \text{eq. 5.10.4}$$

which is the so-called *partial derivative* of $V(x,y)$ for a constant value of $y=y_0$. Subscript y means that we keep y constant when we differentiate with respect to x . The “soft” delta indicates, that V is a function of more variables (here the two variables x and y). When we differentiate V with respect to x , we should keep all other variables constant. The other variables that should be kept constant are listed to the right of the bracket as subscripts. Here, we only have two variables (x,y) . When we differentiate with respect to x , we should keep y constant in the function $V(x,y)$.

When y is constant, the function $V(x,y)$ becomes a function of one variable $V(x)$ as explained above. The derivative in eq.5.10.4 is the slope of this function, which is the increase in $V(x,y)$ or $V(x)$, when we move a unit distance (1 m) in the x -direction.

When we multiply the derivative by the distance dx , we move in the x -direction, we therefore find *the increase in $V(x,y)$ or $V(x)$, when we move dx in the x -direction*:

$$\left(\frac{\partial V}{\partial x}\right)_y dx \text{ eq. 5.10.5}$$

When we move along $\vec{dy}$ parallel with the y -axis, we keep x at the constant value x_0 . Now the electric potential follows the intersection curve between the purple surface $V(x,y)$ and the partial transparent light blue plane $x = x_0$. The intersection is a curve $V(y)$ for constant $x = x_0$. The slope of this curve is

$$\left(\frac{\partial V}{\partial y}\right)_x \text{ eq. 5.10.6}$$

which is the partial derivative of $V(x,y)$ for a constant value of x .

When we multiply the derivative by the distance dy , we move in the y -direction, we find the increase in $V(x,y)$ or $V(y)$, when we move dy in the y -direction:

$$\left(\frac{\partial V}{\partial y}\right)_x dy \text{ eq. 5.10.7}$$

Fig.5.10.3 shows, that when we move the distance $\vec{dl} = (dx, dy)$ from (x_0, y_0) to $(x_0 + dx, y_0 + dy)$ the electric potential increases the amount dV from $V(x_0, y_0)$ to $V(x_0+dx, y_0+dy)$. The movement $\vec{dl}$ may, as mentioned earlier, be spilt up into two components $\vec{dx}$ and $\vec{dy}$ parallel with the x axis and parallel with the y axis, respectively.

When we move the distance $\vec{dx}$, the electric potential function V is increased by the amount

$$\left(\frac{\partial V}{\partial x}\right)_y dx \text{ eq. 5.10.8}$$

Likewise

$$\left(\frac{\partial V}{\partial y}\right)_x dy \text{ eq. 5.10.9}$$

gives us the increase in V , when we move dy in the y direction.

The sum of these two increases in V is the total increase in V , when we first move dx in the x direction and then move dy in the y direction.

$$dV = \left(\frac{\partial V}{\partial x}\right)_y dx + \left(\frac{\partial V}{\partial y}\right)_x dy \text{ eq. 5.10.10}$$

This is the total increase in V (the total differential of V) when we move the distance $\vec{dl} = (dx, dy)$ from point (x_0, y_0) to point (x_0+dx, y_0+dy) .

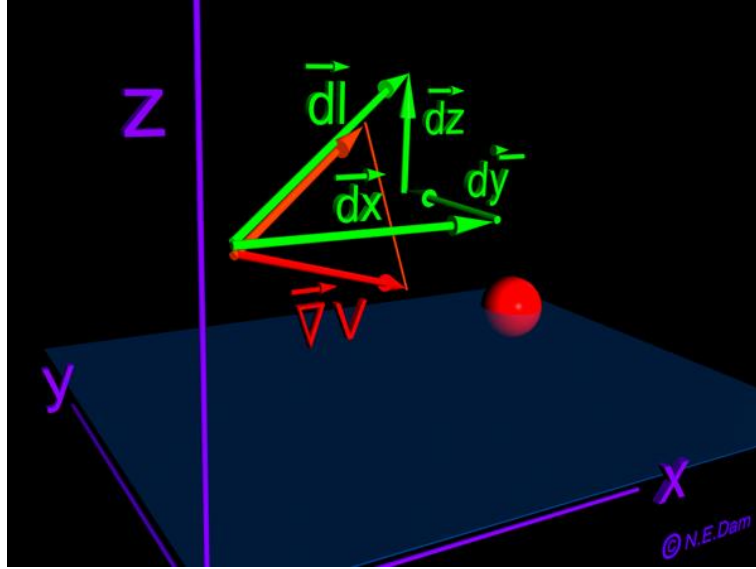


Fig.5.10.4,ref.1 (N.E. Dam ©). This is a 3D electric potential distribution. The increase of V when we move the distance $d\vec{l}$ is the projection of $\vec{\nabla}V$ on $d\vec{l}$ times the length of $d\vec{l}$ (the scalar product of $\vec{\nabla}V$ and $d\vec{l}$).

If we have a 3D situation like in fig.5.10.4 we have a 3D electric potential function $V(x,y,z)$. Now we cannot depict the electric potential as a function of (x,y,z) . It would require a fourth dimension and it would be impossible to depict this in a 3D coordinate system.

Anyway, we want to calculate the increase in V when we move from

$$(x, y, z) \text{ to } (x+dx, y+dy, z+dz)$$

However, we may calculate the total increase in the same way as before:

$$dV = \left(\frac{\partial V}{\partial x}\right)_{y,z} dx + \left(\frac{\partial V}{\partial y}\right)_{x,z} dy + \left(\frac{\partial V}{\partial z}\right)_{x,y} dz \quad \text{eq. 5.10.11}$$

1. The first term on the right side is the increase in V , when we move dx in the x direction.
2. The second term is the increase in V , when we move dy in the y direction.
3. The third term is the increase in V , when we move dz in the z direction.

We now observe, that dV may be written as

$$dV = \left(\left(\frac{\partial V}{\partial x}\right)_{y,z}, \left(\frac{\partial V}{\partial y}\right)_{x,z}, \left(\frac{\partial V}{\partial z}\right)_{x,y} \right) \cdot (dx, dy, dz) = \vec{\nabla}V \cdot d\vec{l} \quad \text{eq. 5.10.12}$$

where $\vec{\nabla}V$ is the so-called gradient (vector) of V

$$\vec{\nabla}V = \left(\left(\frac{\partial V}{\partial x} \right)_{y,z}, \left(\frac{\partial V}{\partial y} \right)_{x,z}, \left(\frac{\partial V}{\partial z} \right)_{x,y} \right) \text{ eq. 5.10.13}$$

and

$$\vec{\nabla} = \left(\left(\frac{\partial}{\partial x} \right)_{y,z}, \left(\frac{\partial}{\partial y} \right)_{x,z}, \left(\frac{\partial}{\partial z} \right)_{x,y} \right) \text{ eq. 5.10.14}$$

is the so-called Del operator.

Eq.5.10.2.8 shows, that the total increase of V when we move the distance $\vec{dl}$ is the projection of $\vec{\nabla}V$ on $\vec{dl}$ times the length of $\vec{dl}$, see also fig.5.10.4.

The direction of $\vec{\nabla}V$ tells us the direction where V increases quickest. The magnitude of $\vec{\nabla}V$ tells us how much V increases per meter, when we move in the direction of $\vec{\nabla}V$. The projection of $\vec{\nabla}V$ on $\vec{dl}$ (shown by orange colour in fig.5.10.4) tells us how much V increases per meter, when we move in the direction of $\vec{dl}$. When we multiply this by the length of $\vec{dl}$ we get the total increase in V , when we move the distance $\vec{dl}$.

Proof of $\vec{E} = -\vec{\nabla}V$

If a test charge is moved from a to b, the electric potential drop is

$$V_{ab} = \int_a^b \vec{E} \cdot d\vec{l} \text{ eq. 5.10.15}$$

If we use eq.5.10.2.8 for dV , this may be written as

$$V_{ab} = V_a - V_b = - \int_a^b dV = - \int_a^b \vec{\nabla}V \cdot d\vec{l} = \int_a^b \vec{E} \cdot d\vec{l} \text{ eq. 5.10.16}$$

which gives

$$\vec{E} = -\vec{\nabla}V \text{ eq. 5.10.17}$$

or

$$E_x = -\frac{\partial V}{\partial x}, E_y = -\frac{\partial V}{\partial y}, E_z = -\frac{\partial V}{\partial z} \text{ eq. 5.10.18}$$

Eq.5.10.3.3 is very important and shows that $\vec{E}$ is the negative gradient vector of the electric potential V .

Now, it is recommended to consider the problems 5.5 and 5.6.

Problems

5.1 The electric potential in the middle of three ions

We consider a molecule formed by the 3 (orange) charges q in fig.5.1a. Then we place an extra (red) ion q_0 in the middle of the molecule, see fig.5.1b. For the extra ion in we have $q_0 = q = e$.

1. Use the method described in chapter 4 to calculate $U_{\text{e pot}}$ of q_0 relative to the other 3 charges and use eq.5.1.3 to calculate the electric potential V at q_0 's position in the middle between the three ions of charge q . (solution 37,4 V)
2. Use eq.5.1.4 to find the same result.

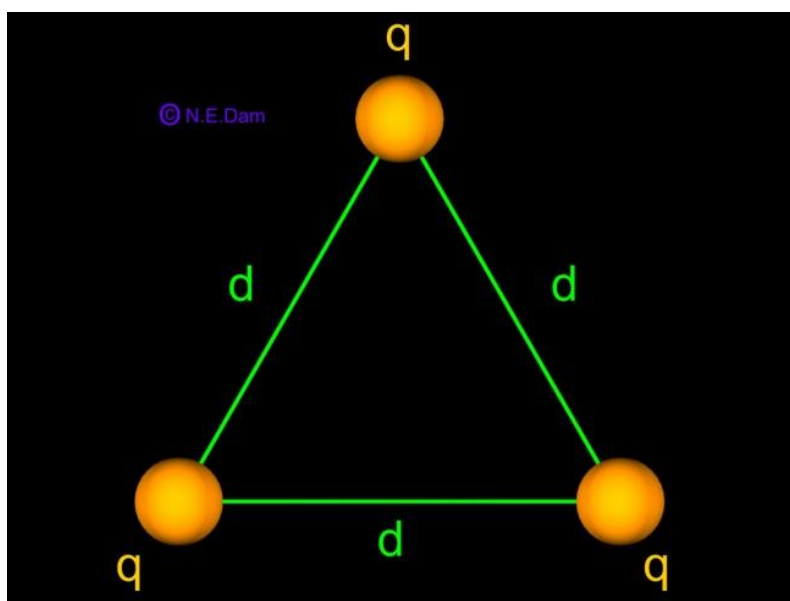


Fig.5.1a, ref.1 (N.E. Dam ©). A triangular molecule with 3 ions (orange) placed in the corners.

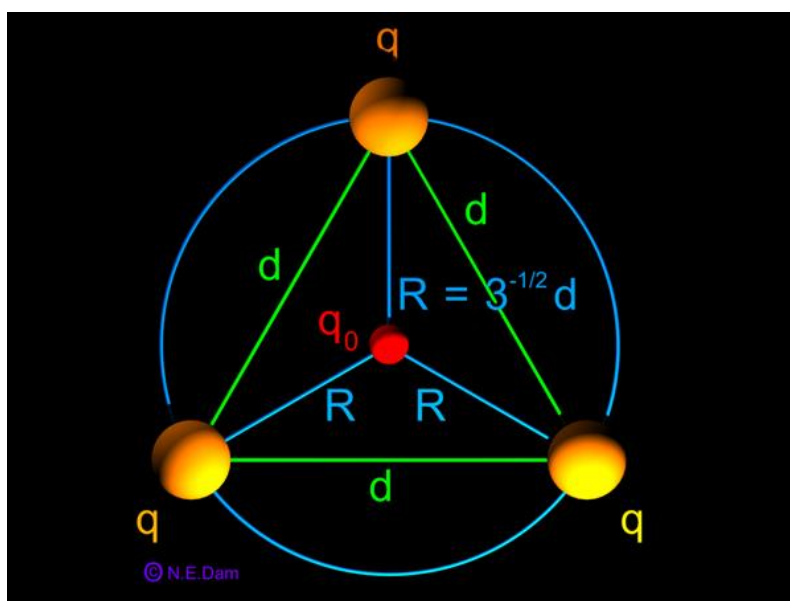


Fig.5.1b, ref.1 (N.E. Dam ©). A molecule with an extra (red) ion placed in the center.

5.2 The electric potential at three points close to a dipole

Fig.5.2a shows a test charge q_0 placed in a mirror plane, where all points have the same distance r to $-q$ and q . The distance between $-q$ and q is d . The mirror plane is perpendicular to the line between $-q$ and q and intersects this line at a distance $d/2$ from $-q$ and q .

1. Show that the electric potential is zero at all points in the plane.

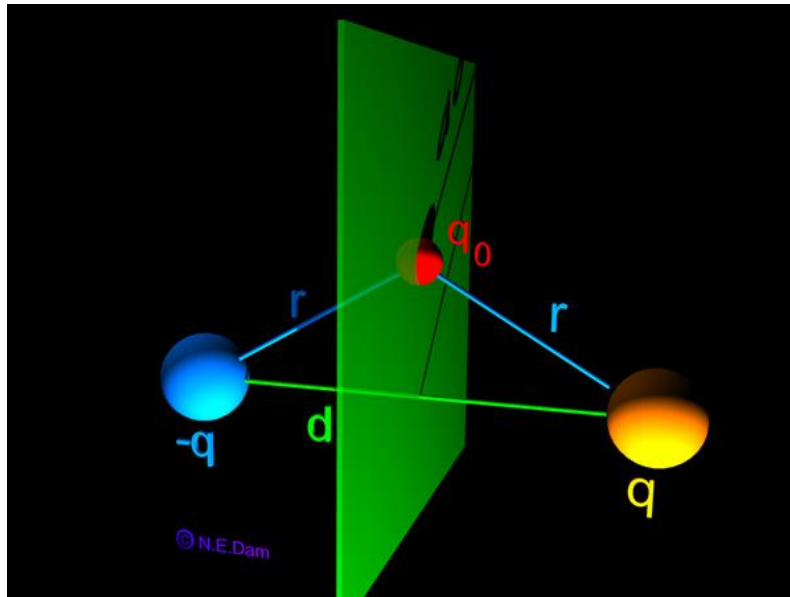


Fig.5.2a, ref.1 (N.E. Dam ©). A test charge q_0 placed in a mirror plane, where all points have the same distance r to $-q$ and q . The distance between $-q$ and q is d .

2. Fig.5.2b shows the test charge q_0 placed “behind” charge q at a distance $d/4$ from q . Now, let $q = 10\text{nC}$ and $d = 10\text{ cm}$.

Calculate the electric potential at the point where we have placed q_0 (2876 V).

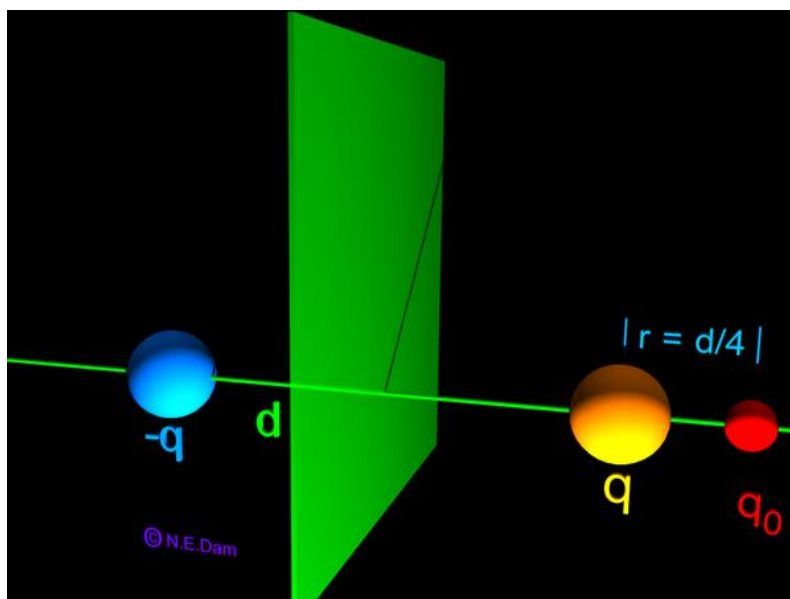


Fig.5.2b, ref.1 (N.E. Dam ©). A test charge q_0 placed behind the charge q at a distance $d/4$ from q .

Fig.5.2c shows the test charge q_0 placed between $-q$ and q at a distance $d/4$ from q .

3. Calculate the electric potential at the point where we have placed q_0 ? (2397 V).

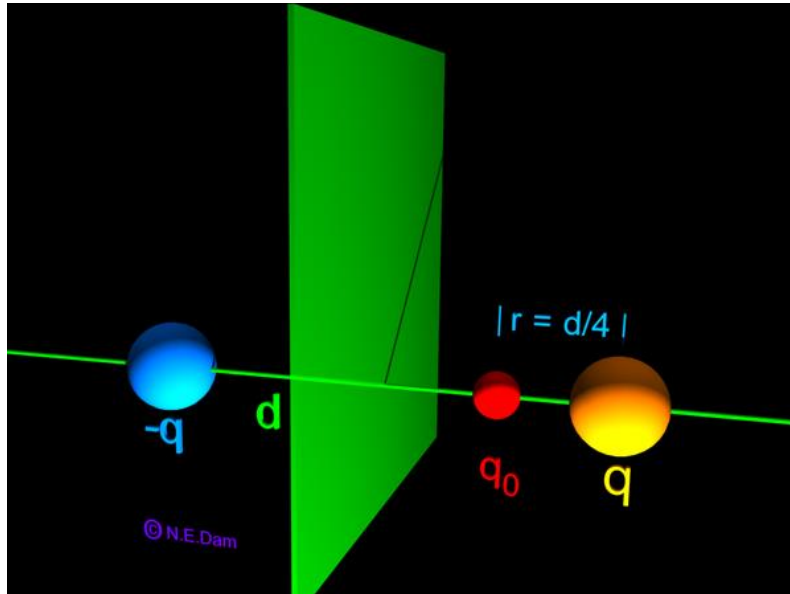


Fig.5.2c, ref.1 (N.E. Dam ©). A test charge q_0 placed between $-q$ and q at a distance $d/4$ from q .

5.3 Acceleration of a proton

A proton moves in a straight line from a to b inside a linear accelerator over a distance of 0,4 m. The electric field is uniform $E = 10 \text{ MV/m}$.

1. What is the force on a proton ?
2. What is the work done by the electric field on the proton ?
3. What is the electric potential drop ?

5.4 Calculation of critical radius for a high voltage equipment.

A high voltage equipment has the voltage 50 kV. The critical E-field in humid air is set to 1 MV/m.

1. What is the critical radius R_c ?
2. What is the distance from the corner to the blue surface in fig.5.4.2 if the radius of curvature is $R_a=40 \text{ mm}$?

5.5 Electric potential and E-field from a spherical charge

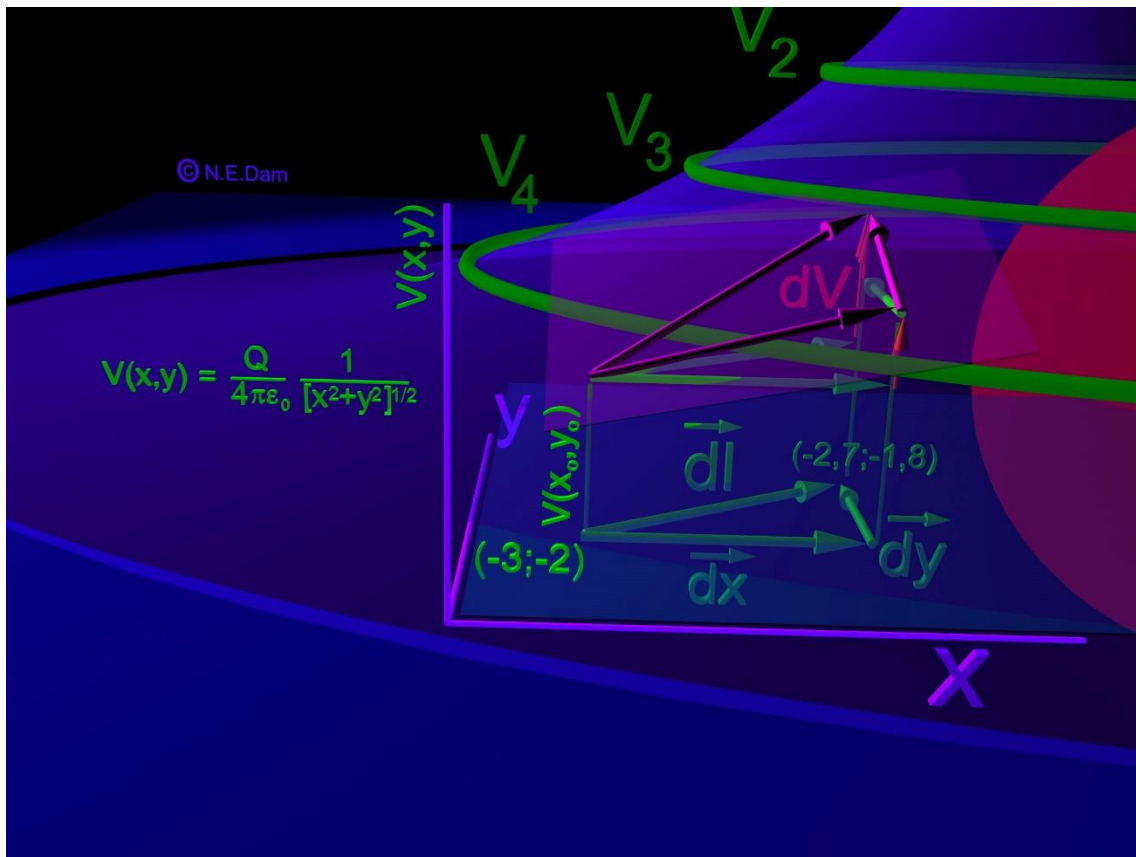


Fig.5.5a, ref.1 (N.E. Dam ©). This figure shows a 3D representation of the 2D electric potential $V(x,y)$ for the spherical charge shown in the right side. We have placed the coordinate system with the origin $(x, y) = (0, 0)$ at the center of the spherical charge (to the right outside the figure).

We consider the $V(x,y)$ shown in fig.5.5a, which is the electric potential from a spherical charge $Q = 1 \text{ nC}$. The theory is described in section 5.3. When we are outside the spherical charge, the electric potential is given by:

$$V(r) = \frac{Q}{4\pi\epsilon_0} \frac{1}{r} \quad \text{eq. 5.3.1}$$

We place the coordinate system with the origin $(x, y) = (0, 0)$ at the centre of the spherical charge. The unit is in meter. Since r^2 equals $x^2 + y^2$, the electric potential may be written as

$$V(x,y) = \frac{Q}{4\pi\epsilon_0} \frac{1}{[x^2 + y^2]^{1/2}} \quad \text{eq. 5.5a}$$

1. Find $\vec{\nabla} V$
2. Find $\vec{E}$
3. Verify the result for E_x and E_y .
4. Find the increase dV in V , when (x, y) changes from $(-3 ; -2)$ to $(-2,7 ; -1,8)$.
5. Please find the real increase ΔV in the electric potential V by substituting the coordinates $(-3 ; -2)$ and $(-2,7 ; -1,8)$ into the function V given above (eq.5.5a)
6. Please compare the result with the result from question 4. What is the difference between the two methods?

Hints (try not to read this before you have tried to solve the problem)

1. The x coordinate of the gradient is given by

$$\left(\frac{\partial V}{\partial x}\right)_y = \frac{Q}{4\pi\epsilon_0} \left(\frac{\partial}{\partial x}\right)_y [x^2 + y^2]^{-\frac{1}{2}} = \frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{2}\right] [x^2 + y^2]^{-\frac{3}{2}} [2x] \Rightarrow$$

$$\left(\frac{\partial V}{\partial x}\right)_y = \frac{-Q x}{4\pi\epsilon_0} [x^2 + y^2]^{-\frac{3}{2}} \text{ eq. 5.5b}$$

Now, verify that

$$\left(\frac{\partial V}{\partial y}\right)_x = \frac{-Q y}{4\pi\epsilon_0} [x^2 + y^2]^{-\frac{3}{2}} \text{ eq. 5.5c}$$

2. Thereby, show that

$$\vec{E} = -\vec{\nabla}V = -\left\{\begin{pmatrix} \left(\frac{\partial V}{\partial x}\right)_y \\ \left(\frac{\partial V}{\partial y}\right)_x \end{pmatrix}\right\} = \frac{Q}{4\pi\epsilon_0} \left\{\begin{matrix} x[x^2 + y^2]^{-\frac{3}{2}} \\ y[x^2 + y^2]^{-\frac{3}{2}} \end{matrix}\right\} \text{ eq. 5.5d}$$

3. In order to verify this result we may use the result for the E-field from section chapter 3:

$$E = \frac{Q}{4\pi\epsilon_0 r^2} \text{ eq. 5.5e}$$

This magnitude should be multiplied by $\cos(\theta)$ to get the x component of the E-field vector, where θ is the angle from the E-field vector to the x-axis.

For a point (x,y) we have

$$r = [x^2 + y^2]^{\frac{1}{2}} \text{ and } \cos(\theta) = \frac{x}{r}$$

This should enable the reader to verify the result for the x component of the E-field. The y component may be verified the same way.

4. We may consider fig.5.5a with $(x_0, y_0) = (-3; -2)$ and $dx = +0,3$, $dy = +0,2$.

Now calculate the value of $\left(\frac{\partial V}{\partial x}\right)_y$ and $\left(\frac{\partial V}{\partial y}\right)_x$ at the point $(x_0, y_0) = (-3; -2)$ and use

$$dV = \vec{\nabla}V \cdot d\vec{l} = \left(\frac{\partial V}{\partial x}\right)_y dx + \left(\frac{\partial V}{\partial y}\right)_x dy \text{ eq. 5.5f}$$

to find the result. In this case the increase dV is calculated from the slope of the potential surface $V(x, y)$ at the point $(x_0, y_0) = (-3; -2)$, i.e. a linear extrapolation, which is only precise if the distance $d\vec{l}$ is relatively small.

5. We have that $\Delta V = V(-2,7; -1,8) - V(-3; -2)$. This is an exact calculation of the real increase in potential, since we take the difference between the potential at the two points.

5.6 Electric potential and E-field from a point charge

In the case of spherical symmetry, as we have in the above exercise, we may use

$$E_r = -\frac{dV}{dr} \text{ eq. 5.6a}$$

We know the electric potential from a point charge Q :

$$V_r = \frac{Q}{4\pi\epsilon_0 r} \text{ eq. 5.6b}$$

1. Please use eq.5.6a and eq.5.6b to find the E field as a function of r :

Note, that we have V and E magnitudes expressed as a function of r. We do not have an expression for vector $\vec{E}$ with the x and y coordinates E_x and E_y .

Test

Max possible score 100p

1! (20p) Definition of the electric potential

- a. Define the electric potential V using U_{epot} and q_0 . Also explain in words.
- b. Please write W_{ab} when an electric force moves q_0 from a to b, and find V_{ab} .
- c. What is V_{aa} (for a closed curve) ?

2 (25p) The potential outside a sphere with the charge Q (a central field)

- a. Find the E-field $E(r)$ in distance r from Q
- b. Find the potential $V(r)$ in distance r from Q .
- c. Please quote the relation between V and E and r .
- d. A sketch may be useful.

3i (10p) Find the potential drop V_{ab} from a to b in a constant E-field and find the alternative definition of the E-field.

4i (12p)

Define the eV by an equation and in words using the work done by the electric force or the drop in potential energy. A sketch may be useful.

5i (5p) Define a potential curve or surface in words.

6. (28p) The gradient

- a. What is the Del operator $\vec{\nabla}$ (defined by an equation) ?
- b. Explain the potential gradient $\vec{\nabla}V$ and write the relation between $\vec{\nabla}V$ and $\vec{E}$.
- c. Derive the relation between $\vec{\nabla}V$ and $\vec{E}$.
NB! You are not supposed to derive the relation $dV = \vec{\nabla}V \cdot d\vec{l}$